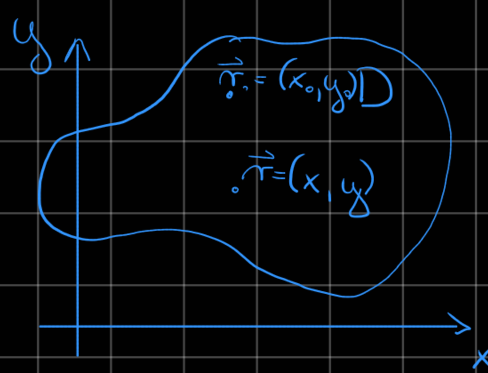
Greenove funkcije

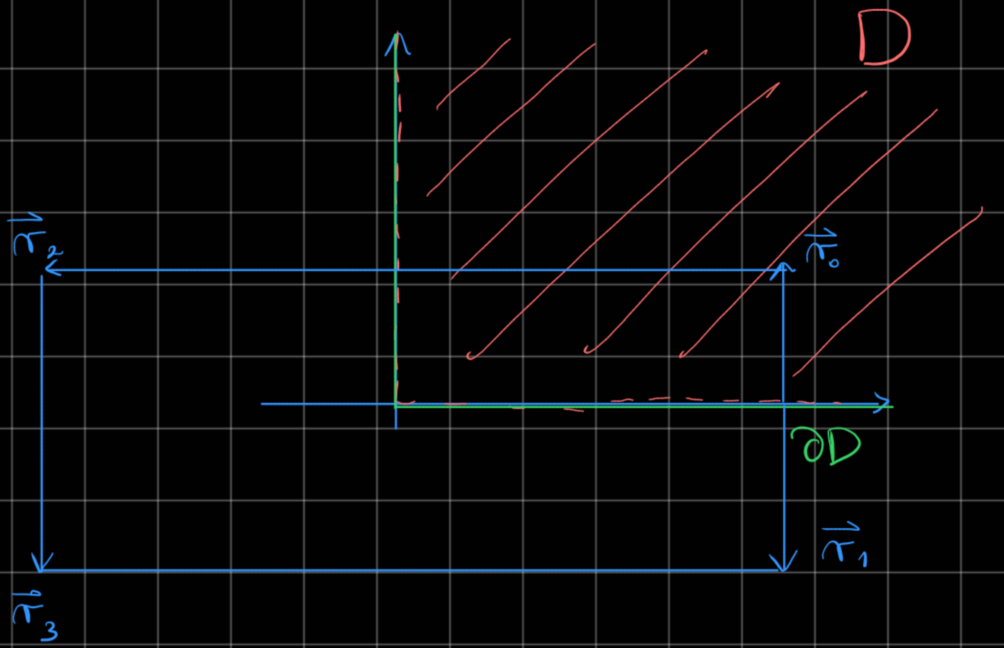


harmonična $\vec{r}$ na D

$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi} \ln |\vec{r} - \vec{r}_0| + \varphi(\vec{r}, \vec{r}_0)$$

$$G(\vec{r}, \vec{r}_0) = 0 \quad \forall \vec{r} \in \partial D$$

P: Poišči Greenovo fjo za $D = \{(x, y) \in \mathbb{R}^2;\ x, y > \emptyset\}$



$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi}\Big(\overset{\emptyset}{\ln|\vec{r} - \vec{r}_0|} - \overset{1}{\ln|\vec{r} - \vec{r}_1|} - \overset{2}{\ln|\vec{r} - \vec{r}_2|} + \underset{3}{\ln|\vec{r} - \vec{r}_3|}\Big)$$

i) $\vec{r} \in$ x-os

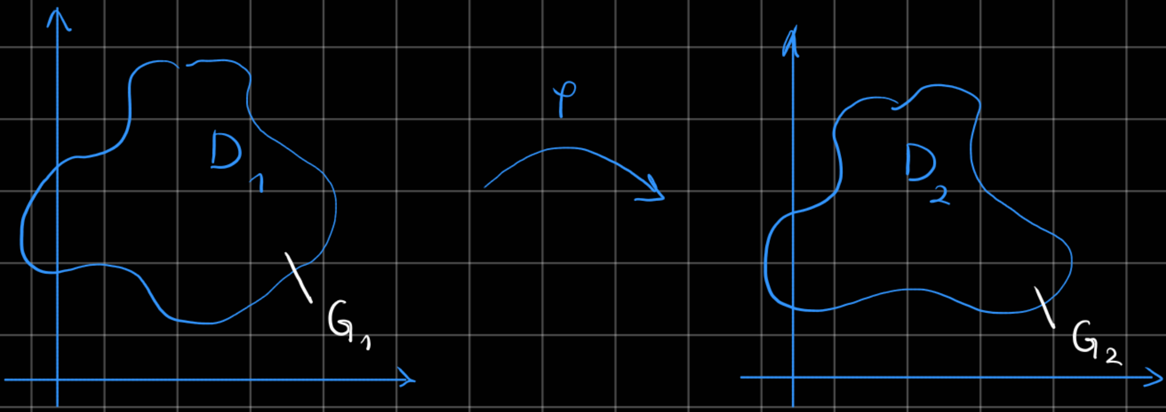
$$(\emptyset - 1) - (2 + 3)$$

ii) $\vec{r} \in y$-os

$$\cancel{\textcircled{0}} - \cancel{\textcircled{1}} - \cancel{\textcircled{2}} + \cancel{\textcircled{3}}$$

Greenove fje in biho l. preslikave

D_1, D_2 dve območji v $\mathbb{C}$ in $\varphi : D_1 \to D_2$ biholomorfna.



$$\vec{r}_1 \longleftrightarrow z$$
$$\vec{r}_0 \longleftrightarrow z_0$$

$$G(z, z_0) = G_2(\varphi(z), \varphi(z_0))$$

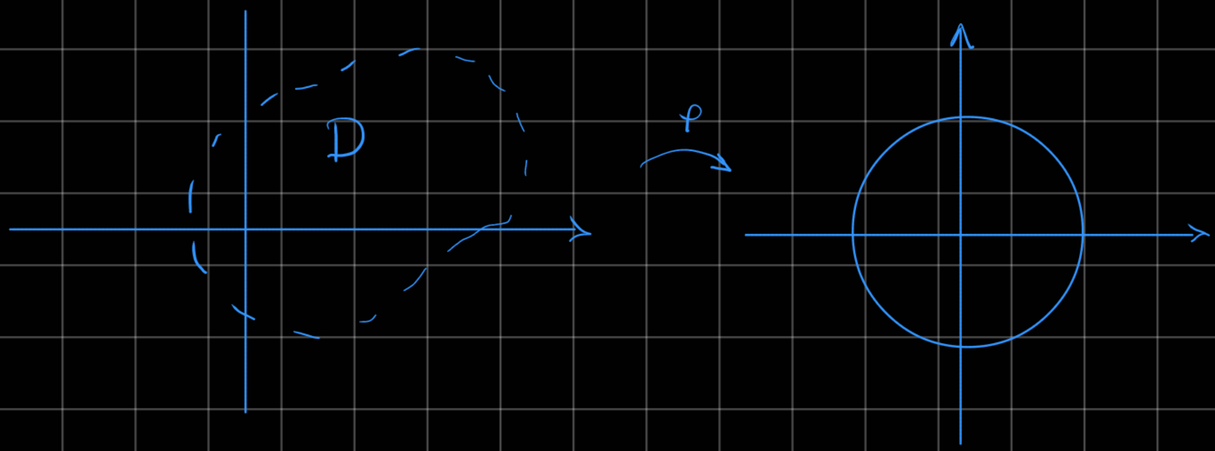
($\varphi : D_1 \to D_2$ in poznamo $G_2 \Rightarrow$ izračunamo G_1)

$$G_1 = \varphi^* G_2$$

Kaj velja?

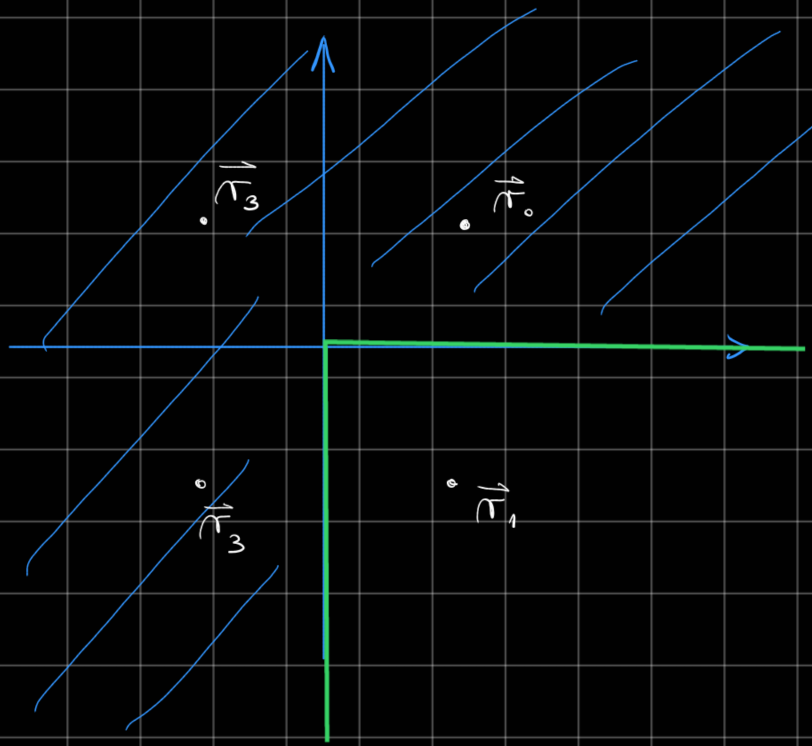
i) (zadušje) Poznamo $G_0(z, z_0)$ za $B(0,1)$

ii) Naj bo D enostavno povezano obm. v $\mathbb{C}$, $D \neq \mathbb{C}$, potem obstaja biholomorfna $\varphi : D \to B(0,1)$



$$G_D(z, z_0) = G_0(\varphi(z), \varphi(z_0))$$

① Poišči Greenovo fjo za $D = \{z \in \mathbb{C} ; z \neq 0 \text{ in } \mathrm{Arg}(z) \in (0, \frac{3\pi}{2})\}$

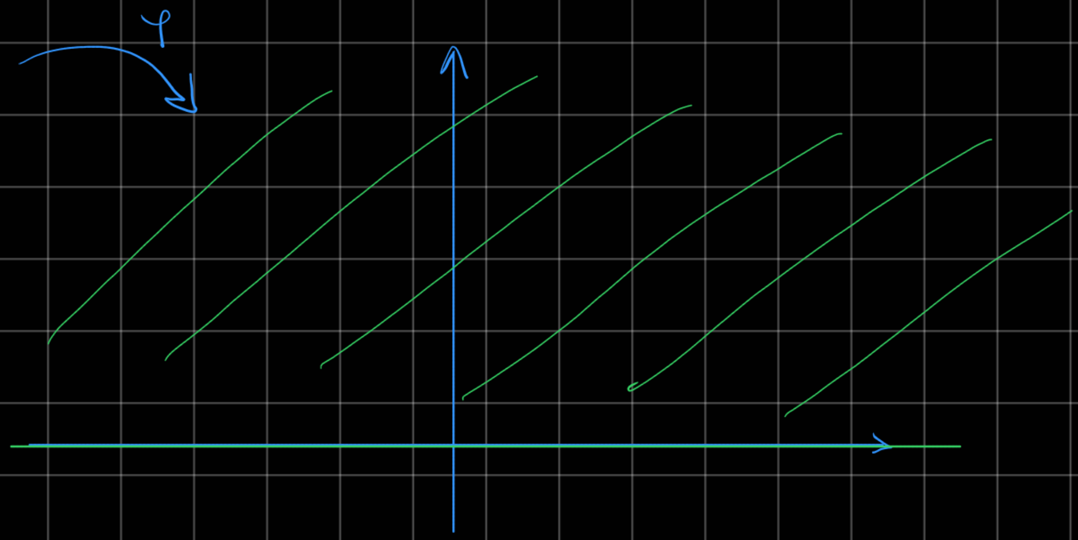


$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi}\left(\ln|\vec{r} - \vec{r}_0| \underbrace{- \ln|\vec{r} - \vec{r}_1| - \ln|\vec{r} - \vec{r}_2| + \ln|\vec{r} - \vec{r}_3|}\right)$$

Ali je harmonična na D?

Ne in tudi definirana ni povsod na D

Ni možno rešiti izključno z zrcaljenji



Preslikamo v zg. polravnino

$$\varphi(z) = z^{\frac{3}{2}}$$

$$z = re^{i\varphi}, \quad \varphi \in \left[0, \frac{3\pi}{2}\right] \Rightarrow \varphi(z) = r^{\frac{2}{3}} e^{i\varphi\frac{2}{3}}$$

$$\Rightarrow \frac{2}{3}\varphi \in (0, \pi)$$

$$G_0 = \frac{1}{2\pi}\left(\ln|\vec{r} - \vec{r}_0| - \ln|\vec{r} - \vec{r}_1|\right)$$

$$G_D = G_0(\varphi(z), \varphi(z_0)) = \frac{1}{2\pi}\left(\ln\left|\vec{r}^{\,\frac{2}{3}} - \vec{r}_0^{\,\frac{2}{3}}\right| - \ln\left|\vec{r}^{\,\frac{2}{3}} - \vec{r}_1^{\,\frac{2}{3}}\right|\right)$$

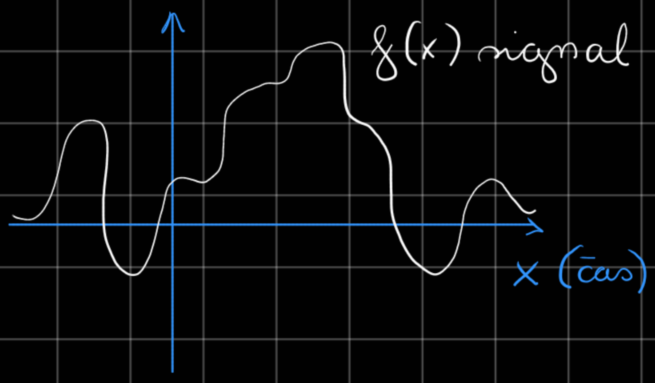
Fourierova transformacija

$$f : \mathbb{R} \to \mathbb{C}$$

$f \xmapsto{\mathcal{F}} \hat{f} \longrightarrow$ Fourierova transformiranka fje f

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

i) Gre za posplošitev Fourierovih vrst



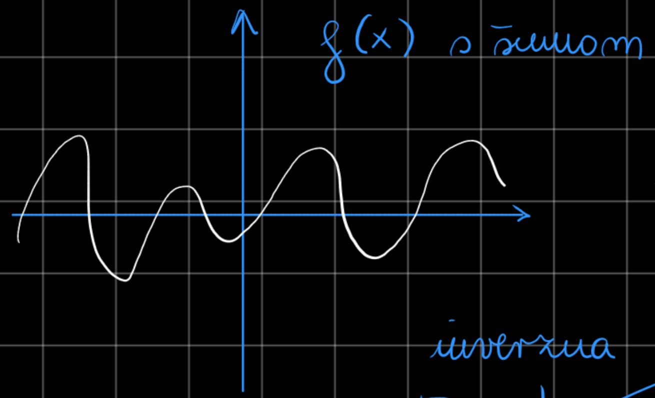
ξ frekvenca

$\Downarrow$

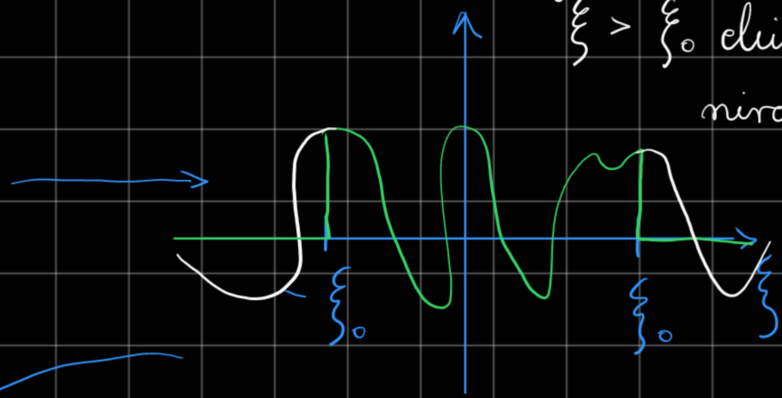
$e^{-i\xi x}$ je signal, ki niha s frekvenco ξ

$\hat{f}(\xi)$ ... delež signala $e^{-i\xi x}$ v $f(x)$

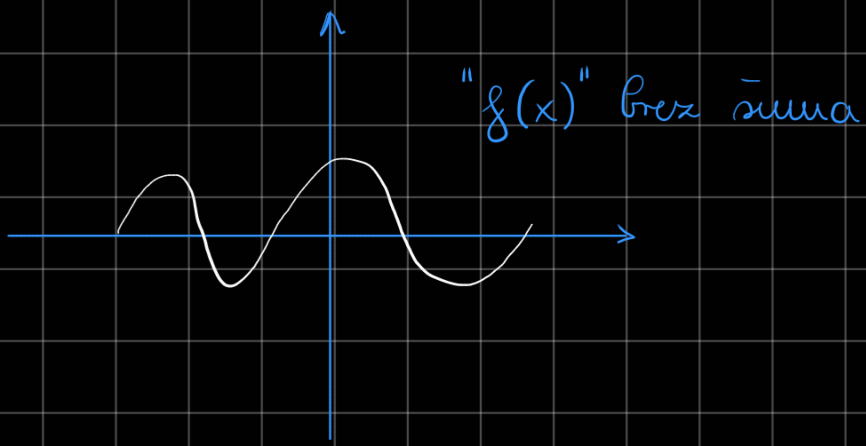
Fizikalno odstranjevanje šuma



// visoki frekvenca $\xi > \xi_0$ elimi= niramo



inverzna Four. tr.



Druge lastnosti $\mathcal{F}$

$$f(x) \rightsquigarrow \hat{f}(\xi)$$

i) $\widehat{f(x)\,e^{ix\xi}}\,(\xi) = \hat{f}\,(\xi - t)$

ii) $\widehat{f(ax)}\,(\xi) = \frac{1}{a}\,\hat{f}\left(\frac{\xi}{a}\right)$

iii) $\widehat{f(x-t)}\,(\xi) = e^{-it\xi}\,\hat{f}(\xi)$

iv) $\widehat{f'(x)}\,(\xi) = i\,\xi\,\hat{f}(\xi)$

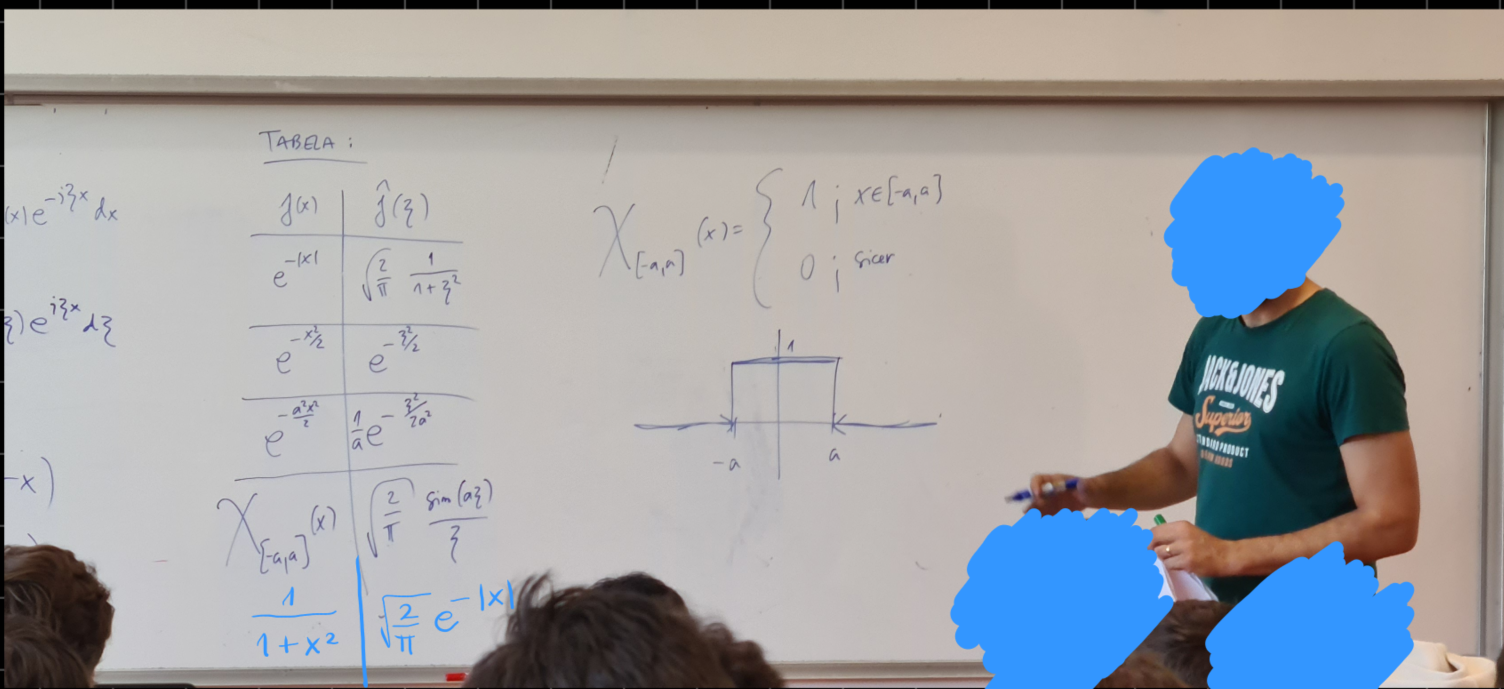
v) inv. Four. tr.

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

vi) $\hat{\hat{f}}(x) = f(-x)$

$\mathcal{F}$ je skoraj sama sebi inverzna



(1) Dana je $f(x) = e^{-ax^2}$ $(a > 0)$. Določi $\hat{f}(\xi)$

$$\hat{f}(\xi) = \widehat{e^{-(ax^2)}}(\xi) = \widehat{e^{-(\sqrt{a})^2 x^2}}(\xi) =$$

$$= \widehat{e^{-\frac{2(\sqrt{a})^2 x^2}{2}}}(\xi) = \widehat{e^{-\frac{(\sqrt{2a})^2 x^2}{2}}}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi}{2a2}}$$

$$\Rightarrow \hat{f}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$$

(2) Dana je $f(x) = e^{-x^2}\cos(2x)$. Določi $\hat{f}(\xi)$.

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\cos(2x) = \frac{e^{2xi} + e^{-2xi}}{2}$$

$$f(x) = e^{-x^2}\frac{e^{2xi} + e^{-2xi}}{2} \Rightarrow \hat{f}(\xi) = \widehat{e^{-x^2}\frac{e^{2xi} + e^{-2xi}}{2}}(\xi)$$

$$= \frac{1}{2}\left(\widehat{e^{-x^2}e^{2xi}}(\xi) + \widehat{e^{-x^2}e^{-2xi}}(\xi)\right) =$$

lastnost i)

$$= \frac{1}{2}\left(\widehat{e^{-x^2}}(\xi - 2) + \widehat{e^{-x^2}}(\xi + 2)\right) \underset{a=1}{\uparrow}=$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} e^{-\frac{(\xi-2)^2}{4}} + \frac{1}{\sqrt{2}} e^{-\frac{(\xi+2)^2}{4}}\right)$$

$$(\xi - 2)^2 = \xi^2 - 4\xi + 4$$

$$(\xi + 2)^2 = \xi^2 + 4\xi + 4$$

$$= \frac{\sqrt{2}}{4} e^{-\frac{\xi^2}{4}-1}\left(\frac{e^{-\xi}+e^{+\xi}}{2}\right) = \frac{\sqrt{2}}{2} e^{-\frac{\xi^2}{4}-1} \operatorname{ch}(\xi)$$

$$= \frac{1}{\sqrt{2}\,e} e^{-\frac{\xi^2}{4}} \operatorname{ch}(\xi)$$

Kaj pa

$$\widehat{e^{-x^2}e^{i2x}}(\xi) = \widehat{e^{-x^2}}(\xi-2) = \frac{1}{\sqrt{2}} e^{-\frac{(\xi-2)^2}{4}}$$

Zanima nas $\frac{1}{1+x^2}$

$$\widehat{e^{-|x|}}(\xi) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2} \quad / \wedge$$

$$e^{-|-x|} = \sqrt{\frac{2}{\pi}} \widehat{\frac{1}{1+\xi^2}}(x)$$

$$\widehat{\frac{1}{1+\xi^2}}(x) = \sqrt{\frac{\pi}{2}} e^{-|x|}$$

③ Dana je $f(x) = \frac{e^{ix}}{x^2+2x+5}$. Določi $\hat{f}(\xi)$

$$\hat{f}(\xi) = \widehat{\frac{e^{ix}}{x^2+2x+5}}(\xi) = \widehat{\frac{e^{ix}}{x^2+2x+4+1}}(\xi) = \widehat{\frac{e^{ix}}{(x+1)^2+4}}(\xi) =$$

$$= \widehat{\frac{1}{(x+1)^2+4}}(\xi-1) = \frac{1}{4} \widehat{\frac{1}{1+\frac{(x+1)^2}{4}}}(\xi-1)$$

lastnost iii) $f(x+1)$

$t=-1$

$$= \frac{1}{4} \widehat{\frac{1}{1+\frac{x^2}{4}}}(\xi-1)\, e^{i(\xi-1)} =$$

$f\left(\frac{x}{2}\right)$ $\quad a = \frac{1}{2}$

$$= \frac{1}{4} \frac{1}{\frac{1}{2}} \widehat{\frac{1}{1+x^2}}\left(\frac{\xi-1}{2}\right) e^{i(\xi-1)} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{-|2\xi-2|} e^{i(\xi-1)}$$

Konvolucija

$f, g: \mathbb{R} \to \mathbb{C}$ $(f(x), g(x))$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Zanima nas podobnost f, g

$$\int_{-\infty}^{\infty} f(x) g(x)\, dx$$

$\to f \approx -g \to$ velika neg. št.

$\to f \approx g \longrightarrow$ velika poz. št.

$$\boxed{\widehat{f * g}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi)}$$

Obratno: $\boxed{\widehat{fg} = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{g}}$ $/\wedge$

$$\hat{\hat{\widehat{fg}}} = \frac{1}{\sqrt{2\pi}} \widehat{f * g}$$

$$fg(-x) = f(-x) g(-x) = \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}\, \hat{\hat{f}}(x)\, \hat{\hat{g}}(x) = f(-x)\, g(-x)$$

4 Poišči $f(x)$ za katere je

$$\underbrace{\int_{-\infty}^{\infty} f(t)\, f(x-t)\, dt}_{(f * f)(x)} = \frac{1}{1+x^2}$$

$$\sqrt{2\pi}\ \hat{f}^2(\xi) = \sqrt{\frac{\pi}{2}}\ e^{-|\xi|}$$

$$\hat{f}(\xi) = \pm\sqrt{\frac{1}{2}\, e^{-|\xi|}}$$

$$\hat{f}(\xi) = \pm\frac{1}{\sqrt{2}}\, e^{-\frac{|\xi|}{2}}$$

$$\shortparallel$$

Iščemo $f(x)$, da velja $\hat{f}(\xi)$?

Predpriprave $\quad \widehat{\frac{1}{1+(ax)^2}}(\xi) = \frac{1}{a}\, \widehat{\frac{1}{1+x^2}}\left(\frac{\xi}{a}\right) = \boxed{\frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-|\frac{\xi}{2}|}}$

Ugannimo

$$\square \longrightarrow e^{-|\frac{\xi}{2}|}$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}}\ \frac{1}{1+(2x)^2} \longrightarrow \frac{1}{2}\sqrt{\frac{\pi}{2}}\ e^{-|\frac{\xi}{2}|}$$

$$\Rightarrow f(x) = \pm\frac{1}{2}\,\frac{2\sqrt{2}}{\sqrt{\pi}}\,\frac{1}{1+(2x)^2} =$$

$$= \pm\frac{2}{\sqrt{\pi}}\,\frac{1}{1+4x^2} = f(x)$$