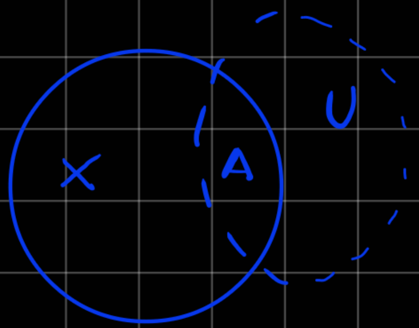
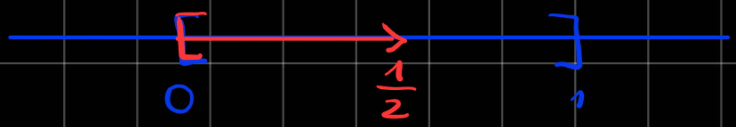
Množica v $\mathbb{R}^n$

Relativna odprtost množica: $X \subseteq \mathbb{R}^n$, $A \subseteq X$

A je rel. odp. mn. v X, če velja $A = X \cap U$, U odp v $\mathbb{R}^n$



P: Ali je $[0, \frac{1}{2})$ odp. mn. v $[0,1)$?

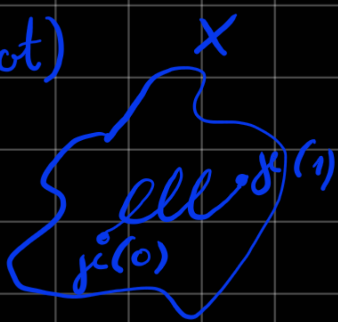


$[0, \frac{1}{2})$ je odprta v $[0,1)$, saj $[0, \frac{1}{2}) = [0,1) \cup (-2, \frac{1}{2})$

Povezanost množica

$X \subseteq \mathbb{R}^n$

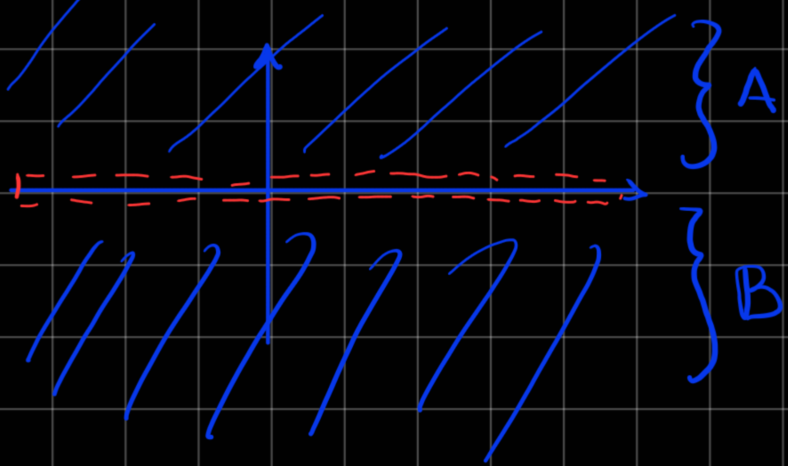
i) X je povezana, če $\forall x, y \in X \; \exists$ zvezna (pot) $\gamma : [0,1] \to X$, $\gamma(0) = x$ in $\gamma(1) = y$



ii) X ni povezana, če lahko X zapišemo kot $X = A \cup B$, kjer sta A, B neprazni, disjunktni in odp. mn. v X (sicer je X povezana)

iii) Če je X s potmi povezana $\Rightarrow$ X je povezana

P: $X = \mathbb{R}^2 - \{\mathbb{R} \times \{0\}\}$

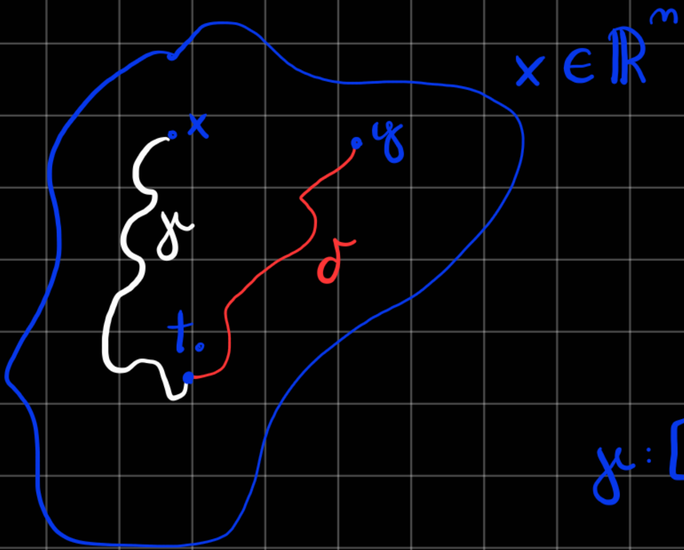


A, B sta neprazni odprti, disj v $\mathbb{R}^2$ in

$X = A \cup B$, X ni povezani

2) Naj bo $X \subseteq \mathbb{R}^n$, $x_0 \in X$. Denimo, da ima X lastnost, da $\forall x \in X$ $\exists$ zvezna pot $\gamma : [0,1] \to X$ od x_0 do x

Pokaži, da je X povezana s potmi:



$x, y \in X$

$\Rightarrow$ iščemo pot Γ od x do y

$\gamma : [0,1]$ pot od x_0 do x $(\gamma(t))$

$\Rightarrow$ pot $x \to x_0$: $\bar{\gamma}(t) = \gamma(1-t)$ zv.

$$\Gamma(t) = \begin{cases} \gamma(1-2t)\ ; & t\in[0,\frac{1}{2}] \\ \delta(2t-1)\ ; & t\in(\frac{1}{2},1] \end{cases}$$ namesto t vstavi $0, 1$

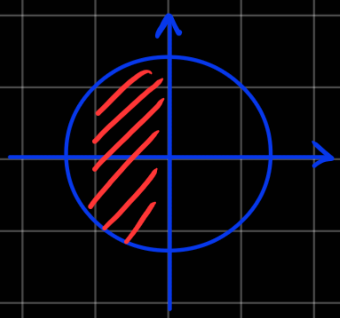
$$\Rightarrow \Gamma(t) = [0,1] \longrightarrow X$$

zveznost: $\lim\limits_{t\uparrow\frac{1}{2}} \Gamma(t) = x_0 = \lim\limits_{t\downarrow\frac{1}{2}} \Gamma(t) = \Gamma(\frac{1}{2})$

3 Dana je mn.

$$K = \{(x,y)\in\mathbb{R}^2;\ x^2+y^2\leq 1 \text{ in } x\leq 0\}$$
$$\cup \{(x,y)\in\mathbb{R}^2;\ x^2+y^2=1\}$$

Pokaži, da je kompaktna in povezana s potmi



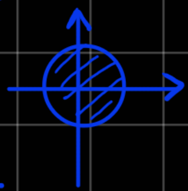
K kompaktna:

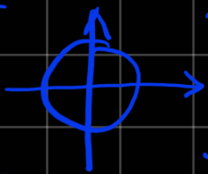
↳ omejena ✓ $K\subseteq B((0,0),2)$

↳ zaprta

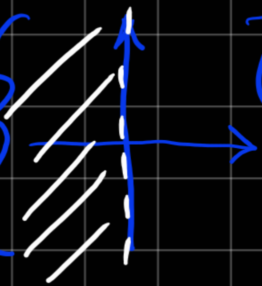
Lahko vzamemo več manjših kompaktnih mn. in iz tega sestavimo K

$f_1 : \mathbb{R}^2 \to \mathbb{R}$ $\quad f_1(x,y) = x^2+y^2$

$A = f_1^{-1}\big(\underbrace{[0,1]}_{\text{zaprta v } \mathbb{R}^n}\big) = \{$  $\} \Rightarrow A$ je zaprta

$B = f^{-1}\big(\underbrace{\{1\}}_{\text{zaprta}}\big) = \{$  $\} \Rightarrow$ zaprta

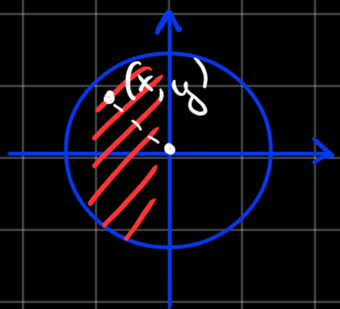
$f_2 : \mathbb{R}^2 \to \mathbb{R} \quad f_2(x,y) = x$

$C = f_2^{-1}\big((-\infty, 0]\big) = \{$  $\} \Rightarrow$ zaprta

$$K = \overbrace{\underbrace{(A \cap C)}_{\text{zaprta}} \cup B}^{\text{zaprta}}$$

Povezanost s potmi

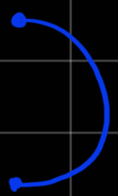
γ



i) $(x,y) \in$ 

$$\gamma(t) = \big((x,y) - (0,0)\big)t + (0,0) =$$
$$= (x,y)t$$

$$\gamma(t) = (tx, ty)$$

ii) $(x,y) \in$  $\qquad \delta(t) = (\sin(t), \cos(t))$

$$t \in [0, \varphi_0]$$

$$\Rightarrow t \in [0,1]$$

$$\delta(t) = (\sin(t\varphi_0), \cos(t\varphi_0))$$

Op.: Kaj je razlika med povezanostjo in -||- s potmi?

i) Pogosto je to dvoje enako (X odp. in povezana $\Rightarrow X$ povezana s potmi)

ii) X ni povezan $\Rightarrow X = A \cup B \Rightarrow f: X \to \mathbb{R}$

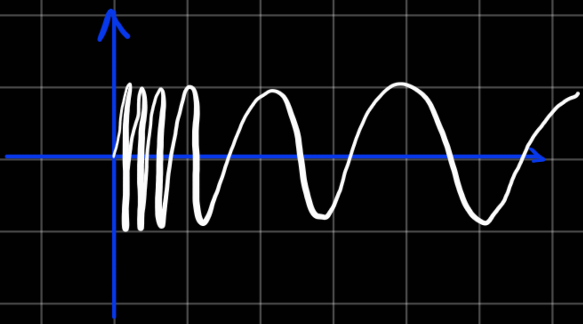
$$f(x) = \begin{cases} f_A(x); & x \in A \\ f_B(x); & x \in B \end{cases}, \quad f_A \neq f_B$$

Če pa X ni povezana s potmi $\Rightarrow$ to ni nujno možno

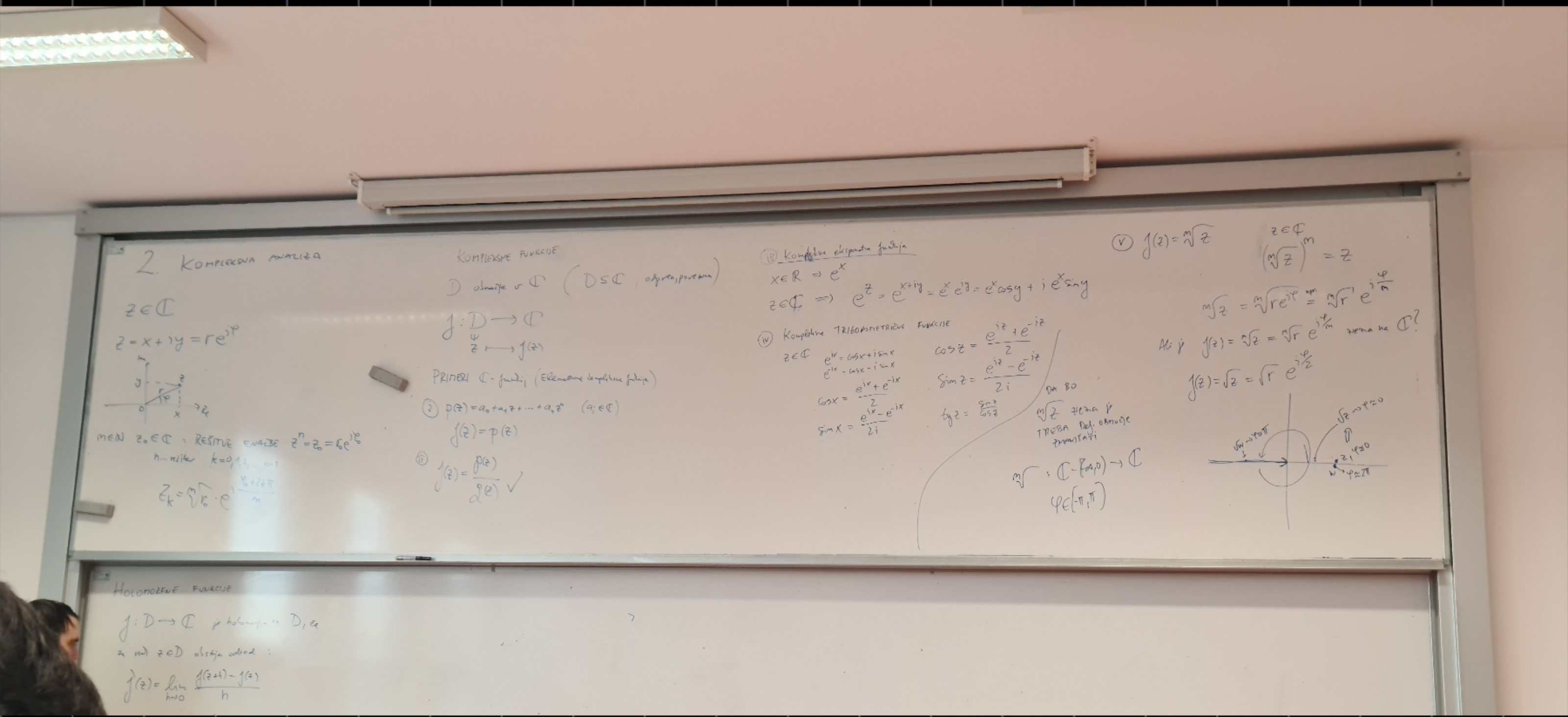
$$X = A \cup B \quad f: X \to \mathbb{R}, \quad f(x) = \begin{cases} f_A(x); & x \in A \\ ?, & x \in B \end{cases}$$

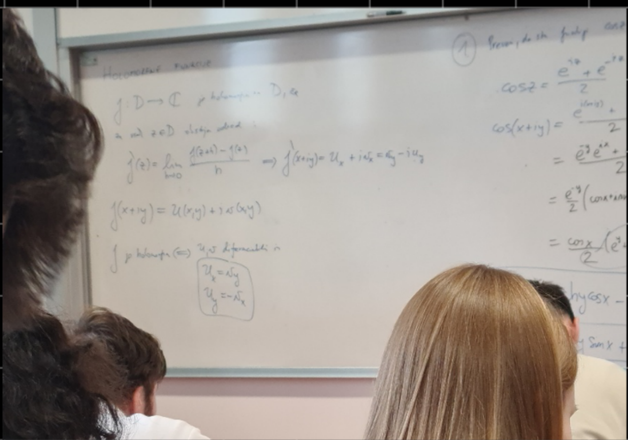
P: $X = \{(x, \sin \frac{1}{x}); x \in (0,\infty)\} \cup \{(0,y); y \in [-1,1]\}$

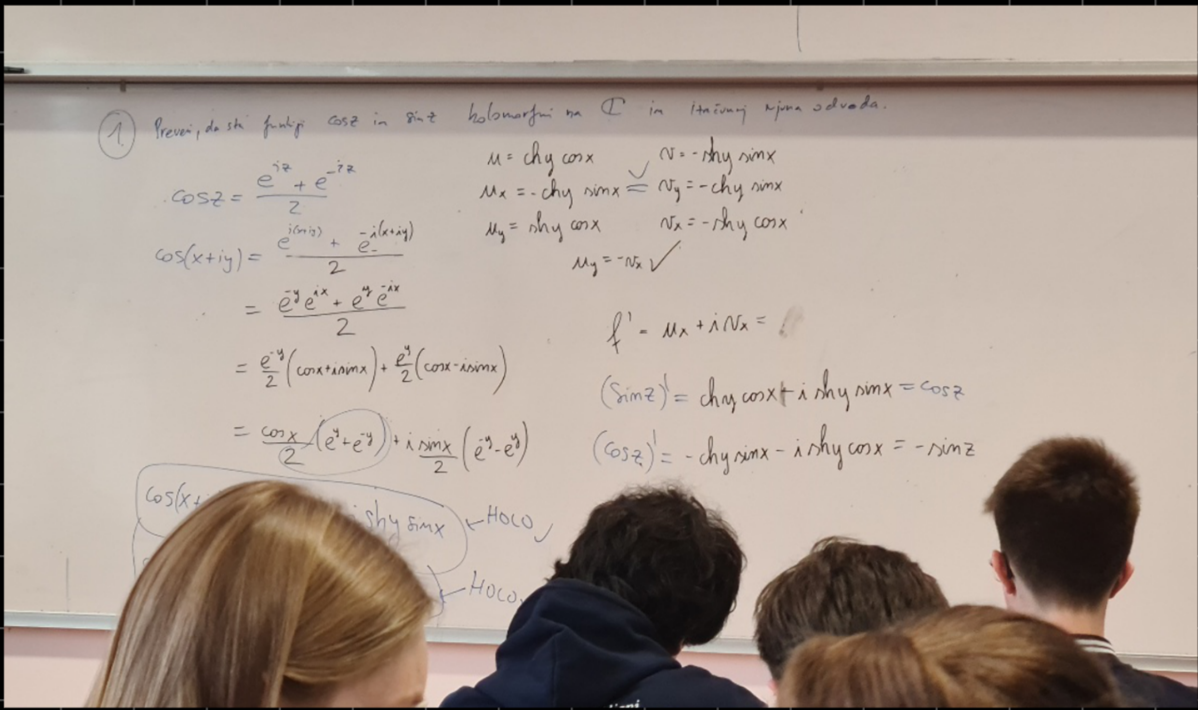
Varšavski lok ni povezan s potmi



$$f(x)=\begin{cases}1; x\in A\\ 1; x\in B\end{cases}$$







2. Dana je fja: $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ s predpisom

$$f(x+iy) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Ali je f holomorfna na $\mathbb{C} \setminus \{0\}$? Zapiši predpis za f direktno s spr. z

$$u_x = v_y = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad u_y = -v_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$f(z) = ? \qquad f(x+iy) = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

Op.: splošen princip, ki reši takšne naloge?

$$f(x+iy) = u(x,y) + iv(x,y) \Rightarrow f(z)$$

$$z = x + iy$$
$$\bar{z} = x - iy$$
$$\left.\begin{array}{l} z = x+iy \\ \bar z = x - iy \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = \dfrac{z+\bar{z}}{2} \\[2ex] y = \dfrac{z-\bar{z}}{2} \end{array}\right\}$$

vstavimo v $f(x+iy)$

$\downarrow$ holomorfne fje

$f \neq f(\bar{z})$