Holomorfne fje

$D \subseteq \mathbb{C}$

$f : D \to \mathbb{C}$

$f(x+iy) = u(x,v) + i\, v(x,y)$

f je hol. na $D \iff$ u, v diferencialni, velja Cauchyjeve-R enačbe

1 Naj $f : D \to \mathbb{C}$ hol. fja. Ali je $F(z) = f(\bar{z})$ hol. fja?

$F(z) = f(\bar{z}) = x - iy$

$f(x+iv) = u(x,y) + i\, v(x,y)$

$$F(x+iy) = \underbrace{u(x,-y)}_{\parallel} + i\,\underbrace{v(x,-y)}_{\parallel}$$

$F(x+iy) = U(x,y) + i\,V(x,y)$

Ali za U, V veljajo CR enačbe, pri čemer predpostavljamo, da CR enačbe veljajo za u, v?

$U_x(x,y) = u_x(x,-y)$ $\qquad V_x(x,y) = v_x(x,y)$

$U_y(x,y) = u_y(x,-y)(-1)$ $\qquad V_y(x,y) = -v_y(x,-y)$

Vemo $\quad u_x = v_y$

$u_y = -v_x$

$$U_x = V_y \qquad\Longleftrightarrow\qquad u_x = -v_y$$
$$U_y = -V_x \qquad\qquad\qquad u_y = v_x$$

F je hol $\Leftrightarrow$ $u_x = u_y = 0$ in $u_y = v_x = 0$, $u, v =$ konst.

$\hookrightarrow$ ko je mali f konst.

2 Naj bo $f: D \to \mathbb{C}$ hol. in $F(z) = \overline{f(\overline{z})}$. Ali je F hol.?

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$F(x+iy) = \underbrace{u(x,-y)}_{U(x,y)} \underbrace{-\,iv(x,-y)}_{+\,iV(x,y)}$$

$$U_x(x,y) = u_x(x,-y)$$
$$\parallel$$
$$V_y(x,y) = -v_y(x,-y)(-1) = v_y(x,-y)$$

// 1. CR enačba drži

$$U_y(x,y) = -u_y(x,-y)$$
$$V_x(x,y) = -v_x(x,-y)$$

$$U_y(x,y) = -V_x(x,y)$$

F je holo. fja

Kaj velja?

- Če je $f(x+iy)$ holo. na D, potem veljajo CR enačbe

$$u_{xx} + u_{yy} = \emptyset \Rightarrow u \text{ harmonična fja}$$

analogno v

- če je $u: D \to \mathbb{R}$ harmonična fja [in D enostavno povezano → 2D brez lukenj], potem $\exists v: D \to \mathbb{R}$, da je $f(x+iy)$ holo. na D

(3) Dana naj bo $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = x^3 - 3xy^2$. Določi $v: \mathbb{R}^2 \to \mathbb{R}$ tako, da bo $f(x,y)$ holo. na $\mathbb{C}$

$$\left.\begin{aligned} u_{xx} &= 6x \\ u_{yy} &= -6x \end{aligned}\right\} \Rightarrow 0 \quad u \text{ je harmonična}$$

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy = -v_x$$

$$\left.\begin{aligned} v &= +3x^2y + C(y) \\ v &= 3x^2y - y^3 + D(x) \end{aligned}\right\} \Rightarrow 0 = 0 + C(y) + D(x) + y^3$$

$$v_y = 3x^2 + C'(y) = -3y^2 + 3x^2$$

$$C(y) = -y^3 + D$$

$$v = 3x^2 - y^3 + D$$

4 Dana naj bo $u: \mathbb{R}^2 \to \mathbb{R}$ $u(x,y) = e^x(\cos(ky) + \sin(ky))$

Določi $v: \mathbb{R}^2 \to \mathbb{R}$ tako, da bo u realni del neke hol. fje in to funkcijo tudi določi.

$$u_x = e^x(\cos(ky) + \sin(ky)) = u_{xx} = v_y$$

$$u_y = e^x k(\cos(ky) - \sin(ky)) = -v_x$$

$$u_{yy} = -e^x k^2(\sin(ky) + \cos(ky)) = -u k^2$$

$$\Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow k = \pm 1$$

$$v = e^x(\sin(y) - \cos(y)) + C(y)$$

$$v_y = e^x(\sin(y) + \cos(y)) + C'(y) = e^x(\cos(y) + \sin(y))$$

$$\Rightarrow C'(y) = 0 \Rightarrow C(y) = 1$$

$$f(x,y) = u(x,y) + iv(x,y) + iD$$

$$= e^x e^{iy} - i e^x e^{iy} + iD$$

$$f(z) = e^z(1-i) + iD$$

5 Naj bo $f: D \to \mathbb{C}$ holo., tako da $|f(z)| = \text{konst}$. Pokaži, da je $f = \text{konst}$.

Op.: $f: S^1 \to \mathbb{C}$

$$f(z) = z^2$$

$$|z| = 1 \Rightarrow |f(z)| = |z^2| = |z|^2 = 1$$

$$u^2 + v^2 = C^2$$

i) $C = 0 \Rightarrow u, v = 0 \Rightarrow f(z) = 0$

ii) $C \neq 0$

$$u = C \sin(\varphi(x,y))$$

$$v = C \cos(\varphi(x,y))$$

$$u_x = C \cos(\varphi(x,y)) \frac{d\varphi}{dx} \qquad v_x = -C \sin(\varphi(x,y)) \frac{d\varphi}{dx}$$

$$u_y = C \cos(\varphi(x,y)) \frac{d\varphi}{dy} \qquad v_y = -C \cos(\varphi(x,y)) \frac{d\varphi}{dy}$$

$u_x = v_y$

$$\cos(\varphi)\frac{d\varphi}{dx} = -\sin(\varphi)\frac{d\varphi}{dy} \quad /\cos(\varphi) \quad /-\sin(\varphi)$$

$$\cos^2(\varphi)\frac{d\varphi}{dx} = -\sin(\varphi)\cos(\varphi)\frac{d\varphi}{dx}$$

$$\frac{d\varphi}{dx} = 0 \qquad \frac{d\varphi}{dy} = 0$$

$u_y = -v_x$

$$\cos(\varphi)\frac{d\varphi}{dy} = \sin(\varphi)\frac{d\varphi}{dx} \quad /\sin(\varphi) \quad /\cos(\varphi)$$

$$\sin^2(\varphi)\frac{d\varphi}{dx} = \sin(\varphi)\cos(\varphi)\frac{d\varphi}{dy}$$

Op.: Kar nekaj podobnih lastnosti velja

$f^{\text{hol.}}: \mathbb{C} \to \mathbb{C}$, či je $|f|$ omejena $\Rightarrow f$ konst.

Liouvilleov izrek

6 Naj bo $f: D \to \mathbb{C}$ holo. fja z lastnostjo, da je

$|f(x+iy)| = e^x(x^2+y^2)$. Določi z!

$$|f(x+iy)| = e^x|z|^2 = |e^z||z|^2 = |z^2e^z| = |f(z)|$$

$$e^z = e^x e^{iy} \Rightarrow |e^z| = e^x$$

$$\left|\frac{f(x)}{x^2e^x}\right| = 1 \Rightarrow \frac{f(x)}{x^2e^x} = c\,,\ |c|=1$$

$$\Rightarrow f(x) = Cx^2e^x$$

Potenčne vrste

$\sum_{n=0}^{\infty} c_n (x-x_0)^n$ je pot. vrsta s središčem v x_0

Za katere $x \in \mathbb{C}$ takšna vrsta konvergira? Vsaka pot. vrsta ima konv. radij $R \geq 0$

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} \qquad \frac{1}{R} = \lim_{n\to\infty}(\sup)\sqrt[n]{c_n}$$

- za $|x-x_0| < R \Rightarrow$ vrsta konv.
- za $|x-x_0| > R \Rightarrow$ div.
- za $|x-x_0| = R \Rightarrow$ odvisno od vrste

7 Izračunaj R

a) $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ $\qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \frac{1}{4}$$

b) $\sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \ldots$

koeficienti $c_n = \begin{cases} 1; & n = 2^k \\ 0; & \text{sicer} \end{cases}$

$\sqrt[n]{|c_n|}$ vsebuje ∞ ničel $\Rightarrow 0$ je središče

$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[2^n]{\underset{c_{2^n}}{\underset{\parallel}{1}}} = 1$

c) $p \in \mathbb{R}$

$\sum_{n=1}^{\infty} n^p z^n$

$R = \lim_{n\to\infty} \frac{n^p}{(1+n)^p} = \lim_{n\to\infty} \Big(\underset{\underset{1}{\Downarrow}}{\frac{n}{n+1}}\Big)^p = 1$

d) $\sum_{n=0}^{\infty} 2^n z^{n!} = z + 2z + 4z^2 + 8z^6$

veliko koef. vrste je 0 $\quad c_n = \begin{cases} 2^k, & k = n! \\ 0; & \text{sicer} \end{cases}$

$c_{n!} = 2^{n!}$

$R = \lim_{n\to\infty} \sqrt[n!]{2^n} = \lim_{n\to\infty} 2^{\frac{1}{(n-1)!}} = 1$

e)