Potenčne vrste

Alt. pristop

Uporaba konv. krit. na vrsti (in ne na koef.)

1) kvocientni kriterij

$$\lim_{n\to\infty} \frac{|c_{n+1}(z-z_0)^{n+1}|}{|c_n(z-z_0)^n|} < 1$$

Kaj velja?

Naj bo $f(z)$ hol. fja na okolici z_0. Potem lahko f razvijemo v potenčno vrsto (Taylor) okoli z_0:

$$f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

kjer ima potenčna vrsta, ki pripada f, konv. radij

$R > 0$

Op. vrsta $\sum_{n=0}^{\infty} n!\, z^n$ ima $R=0 \Rightarrow$ ta vrsta ni Taylor. vrsta nobene holomorfne fje

Znani razvoji

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \qquad z \in \mathbb{C}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

1) Razvij fjo $\sin(z)$ v potenčno vrsto okoli $z_0 = i$

Zg. je okoli $z_0 = 0$. Želimo $\sin(z-i) = (z-i) - \frac{(z-i)^3}{3!} + \frac{(z-i)^5}{5!}$

$$\sin(z) = \sin((z-i)+i) = \sin(z-i)\cos(i) + \sin(i)\cos(z-i)$$

//predpostavimo, da veljajo adicijski izreki

$$= \left((z-i) - \frac{(z-i)^3}{3!} + \frac{(z-i)^5}{5!} - \dots\right)\cos(i)$$
$$\left(1 - \frac{(z-i)^2}{2!} + \frac{(z-i)^4}{4!} - \dots\right)\sin(i)$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \Rightarrow \sin(i) = i\,\mathrm{sh}(1)$$

$$\cos(i) = \operatorname{ch}(1)$$

$$= i\operatorname{sh}(1) + (z-i)\operatorname{ch}(i) - \frac{(z-i)^2}{2!} i\operatorname{sh}(1) - \frac{(z-i)^3}{3!}\operatorname{ch}(1) + \dots$$

(2) Razvij funkcijo $\frac{1}{1-z}$ v potenčno vrsto okoli $z_0 = -1+i$

$$\frac{1}{1-z} = \frac{1}{1-((z+1-i)+(i-1))} = \frac{1}{1-(z+1-i)-i+1} =$$

$$= \frac{1}{2-i-(z+1-i)} = \frac{1}{1-\frac{(z+1-i)}{2-i}} \frac{1}{2-i}$$

Imamo $\frac{1}{A-\square} = \frac{1}{A} \frac{1}{1-\frac{\square}{A}}$

$$= \frac{1}{2-i}\left(1 + \left(\frac{z+1-i}{2-i}\right) + \left(\frac{z+1-i}{2-i}\right)^2 + \dots\right)$$

Op. Razvoji uporabni pri kompleksnih integralih

(3) Denimo, da ima pot. vrsta $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ konv. radij R

Določi konv. radija vrst

$$\sum_{n=0}^{\infty} c_n^2 (z-z_0)^n \quad \text{in} \quad \sum_{n=0}^{\infty} c_n (z-z_0)^{2n}$$

koeficient z indeksom $2n$ ima vrednost c_n

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$$

$$\frac{1}{R_1} = \limsup_{n\to\infty} \sqrt[n]{|c_n^2|} = \limsup_{n\to\infty} \left(\sqrt[n]{|c_n|}\right)^2 =$$

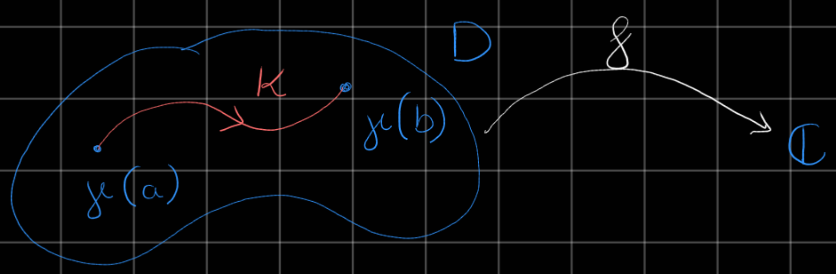
limita podzaporedja k največjemu stekališču

$$= \left(\lim_{n\to\infty} \sqrt[n]{|c_n|}\right)^2 = \frac{1}{R^2}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} \sqrt[2n]{c_n} = \limsup_{n\to\infty} \left(\sqrt[n]{|c_n|}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{R_2}}$$

Kompleksni integral

$f: D \to \mathbb{C}$, f ni nujno holomorfna



K orientirana krivulja v D, parametrizirana z neko

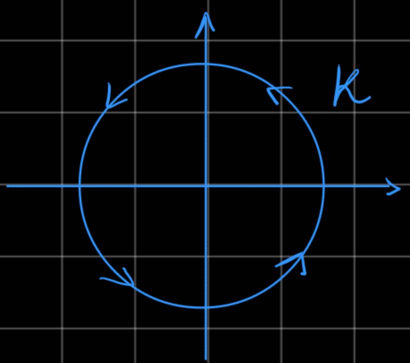
$$\gamma : [a, b] \to D$$
$$t \mapsto \gamma(t)$$

K je orientirana v meri naraščajočega parametra t.

Oznaka: $\int_K f(z)\,dz = (\pm)\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

orientacija krivulje

① Naj bo $n \in \mathbb{Z}$ in $f(z) = z^n$. Naj bo $K = \{z \in \mathbb{C}, |z| = 1\}$ $= \{S^1\}$ poz. orientirana. Izračunaj $\int_K f(z)\,dz$



$\gamma(t) = \cos(t) + i\sin(t) = e^{it}$,

$t \in [0, 2\pi]$

$$\int_K f(z)\,dz = \int_0^{2\pi} (e^{it})^n \cdot ie^{it}\,dt = i\int_0^{2\pi} e^{it(n+1)}\,dt = \begin{cases} 1, & n = -1 \\ e^{it(n+1)}, & \text{sicer} \end{cases}$$

a) $n = -1$

$$\int_K f(z)\,dz = i\int_0^{2\pi} dt = i2\pi$$

b) $m \neq -1$

$$\int_K f(z)\,dz = i\int_0^{2\pi} e^{it(m+1)}\,dt = \frac{1}{m+1}\int_0^{2\pi i(m+1)} e^{u}\,du =$$

$$u = it(m+1)$$

$$du = i(m+1)\,dt$$

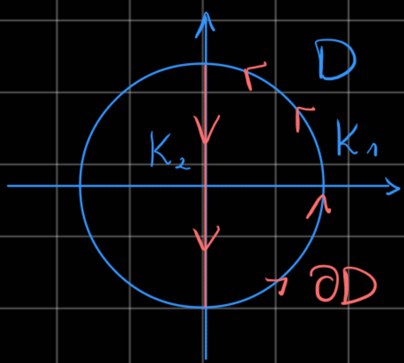
$$\frac{1}{1-m}\left[e^{2\pi i(m+1)} + 1\right] = \frac{1}{1-m}\left[\overbrace{\cos(2\pi(m+1))}^{1} + i\sin(\cancel{2\pi}(m+1))\right.$$

$$\left. -1\right] = \emptyset$$

$$\int_K \frac{dz}{z} = 2\pi i, \text{ drugače je } \emptyset$$

② $f(z) = \bar{z}$

Izračunaj integral $\displaystyle\int_{\partial D^+} f(z)\,dz$, $D = \{z \in \mathbb{C}, |z| < 1 \text{ in } \mathrm{Re}(z) \geq 0\}$



K_1 ... polkrožnica

$K_1: \gamma_1(t) = e^{it}, \; t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

K_2 ... daljica

$K_2: \gamma_2(t) = it, \; t \in [-1, 1]$

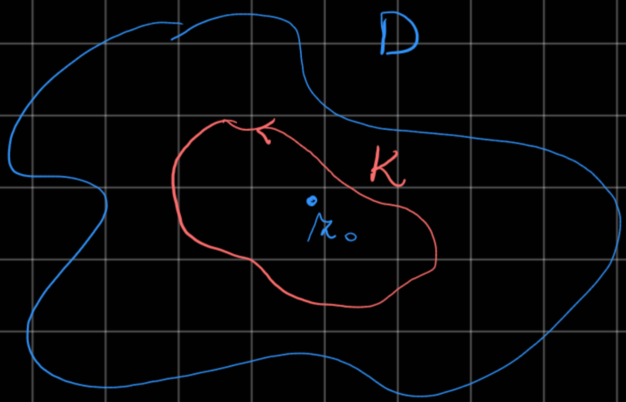
$$\int_{\partial D} f(z)\,dz = \int_{K_1} e^{-it}\, i e^{it}\,dt + \int_{K_2} (-it)\, i\,dt =$$

$$= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt + \int_{-1}^{1} t\,dt = \underline{\underline{i\pi}}$$

Kompleksni integral holomorfne funkcije

$f: D \to \mathbb{C}$ hol. fja

K... sklenjena krivulja v D, poz. orientirana



Velja: 1) $\oint_K f(z)\,dz = 0$

2) Naj $z_0 \in D$ leži znotraj K

$$\frac{1}{2\pi i}\oint_K \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

// uporabimo za hiter izračun kompl. integrala

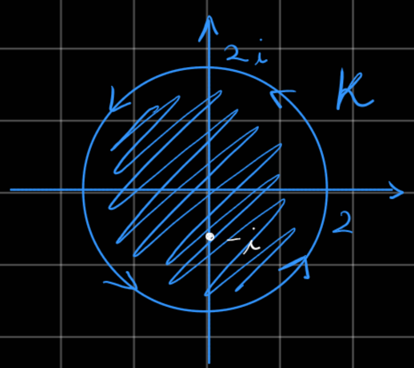
$$\frac{1}{2\pi i}\oint_K \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)\frac{1}{n!}$$

3) $\frac{1}{2\pi i} \oint_K \frac{f(z)}{z - z_0} dz = f(z_0) \; I_K(z_0)$

I_K je indeks K glede na z_0

$$I_K = \begin{cases} 0, & K \text{ ne obkroži } z_0 \\ 2; & K\ 2x \text{ obkroži } z_0 \text{ v + smeri} \end{cases}$$

③ Izračunaj $\int_K \frac{\sin(z)}{z+i} dz$, kjer je $K = \{ z \in \mathbb{C}, |z| = 2 \}$ in + orientirana.



Ali je $\frac{\sin(z)}{z+i}$ holomorfna na območju, ki ga omejuje K?

Ne, v tč. $z_0 = -i$ ni dif.

$\int_K \frac{\sin(z)}{z+i}$, $f(z) = \sin(z)$ je holomorfna

$$= \int_K \frac{f(z)}{z+i} dz = 2\pi i \sin(-i) = 2\pi i \; \frac{e^{i(-i)} - e^{-i(-i)}}{2} = 2\pi \sinh(1)$$

$I_K = 2\pi i$, K enkrat obkroži $z = -i$

(4) $\int_K \frac{z^2}{z-2i}\,dz$, kjer je $K = \{z \in \mathbb{C},\ |z| = 3$ in $+$ orient$\}$

$f(z)$ holomorfna

$$\int_K \frac{z^2}{z-2i} = 2\pi i\,(2i)^2 = -8\pi i$$

$z_0 = 2i$

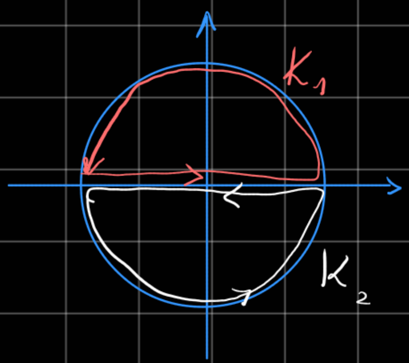
(5) $\int_K \frac{dz}{z^2+9}$, $K = \{z \in \mathbb{C},\ |z| = R\}$, $+$ orient. $R = 2$

$R = 4$

$$\frac{1}{z^2+9} = \frac{1}{(z+3i)(z-3i)}$$

2 singularnosti $z_{01} = 3i$

$z_{02} = -3i$



Za $R = 2$: ne obkroži nobene singularnosti $\Rightarrow \int_K \frac{dz}{z^2+9} = 0$

Cauchyjeva int. formula deluje le za eno obkroženo singularnost, razdelimo K na unijo dveh zank K_1 in K_2.

$$\int_K \frac{dz}{z^2+9} = \int_{K_1} \frac{\frac{dz}{(z+3i)}}{(z-3i)} + \int_{K_2} \frac{\frac{dz}{(z-3i)}}{(z+3i)} = 2\pi i \left[\frac{1}{6i} - \frac{1}{6i}\right] = \emptyset$$