Laurentove vrste

① Razvij $f(z)=\dfrac{1}{z(z-1)(z-2)}$ v Laurent. vrsto (s središčem v 0), ki bo konv. na

a) $0<|z|<1$

b) $1<|z|<2$

c) $|z|>2$

$$\frac{1}{z(z-1)(z-2)}=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}$$

$$\Rightarrow A=\frac{1}{2},\quad B=-1,\quad C=\frac{1}{2}:\quad f(z)=\frac{1}{2}\frac{1}{z}-\frac{1}{z-1}+\frac{1}{2}\frac{1}{z-2}$$

a) $f(z)=\frac{1}{2}\frac{1}{z}+\frac{1}{1-z}+\frac{1}{4}\frac{1}{1-\frac{z}{2}}$

$\left(\frac{1}{1-x}=1+x+x^2+\dots\right)$ $\quad\left|\frac{z}{2}\right|<1$

$|x|<<1$ $\quad |z|<2$

$$=\frac{1}{2}\frac{1}{z}+\underbrace{(1+z+z^2+\dots)-\frac{1}{4}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)}_{\text{konv. za } 0<|z|<1}$$

b) TRIK

$$f(z)=\frac{1}{2}\frac{1}{z}-\overbrace{\frac{1}{z-1}}^{\text{problematic child}}+\frac{1}{2}\frac{1}{z-2}$$

$$= \frac{1}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{4}\frac{1}{1-\frac{z}{2}} = \frac{1}{2z} - \frac{1}{z}\underbrace{\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)}_{\left|\frac{1}{z}\right|<1}$$

$$|z|>1$$

$$-\frac{1}{4}\underbrace{\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)}_{|z|<2}$$

$$\Rightarrow 1 < |z| < 2$$

c) $$f(z) = \frac{1}{2}\frac{1}{z} + \frac{1}{1-z} + \frac{1}{4}\frac{1}{\frac{z}{2}\left(1-\frac{2}{z}\right)} =$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

$$+\frac{1}{2z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots\right)$$

$$\Rightarrow |z|>2$$

2) Izračunaj prve 4 člene L. vrste fje $f(z) = \frac{1}{e^z-1}$ okoli 0. Ugotovi, kje dana vrsta konvergira.

$$f(z) = \frac{1}{e^z-1} \underset{\uparrow}{=} \frac{1}{1+z+\frac{z^2}{2}+\frac{z^3}{6}+\dots-1} = \frac{1}{z}\frac{1}{1+\underbrace{\left(\frac{z}{2}+\frac{z^2}{6}+\dots\right)}_{\text{konv. } z=0 \Rightarrow 0}}$$

e^x razvijemo

z je zelo blizu $0 \to \Rightarrow \exists r>0$

$$|z|<r \Rightarrow \left|\frac{z}{2}+\frac{z^2}{6}+\dots\right|<1$$

$$= \frac{1}{z}\left(1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\dots\right)+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\dots\right)^2\right.$$

$$\left.-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\dots\right)^3+\dots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{2}z + \left(-\frac{1}{6} + \frac{1}{4}\right)z^2 + \left(-\frac{1}{24} + 2\cdot\frac{1}{2}\cdot\frac{1}{6} - \frac{1}{8}\right)z^3 + \dots\right)$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + 0\cdot z^2 + \dots \Rightarrow \text{kje konv. } 0 < |z| < ?$$

Def. območje $f(z) = \dfrac{1}{e^z - 1}$

Problematična tč. je $e^z = 1 \Rightarrow z = 2\pi i k, \quad k \in \mathbb{Z}$

$f(z)$ je def. $0 < |z| < 2\pi$

↑

z ni tako zelo blizu 0

Liouvillov izrek in princip identitete

i) L. izr.

Naj bo $f: \mathbb{C} \to \mathbb{C}$ holom., takšna, da $|f(z)| \leq M$ (za nek $M > 0$) $\forall z \in \mathbb{C}$. Potem je f konst.

ii) Pr. ident.

Naj bosta $f, g: D \to \mathbb{C}$ holom. in $A \subseteq D$ s stekališčem.
Če $f|_A = g|_A \Rightarrow f = g$ na D

Op. $A \subseteq D$

- $x \in D$ je stekališče mn. A, č $\forall r > 0$ velja, da je v $B(x, r) - \{x\}$ kakšen el. mn. A

- $A \subseteq D$ je mn. s stekališčem, če $\exists$ v D kakšno stekališče A

③ Naj bo $f: \mathbb{C} \to \mathbb{C}$ holom. fja, z lastnostjo $\operatorname{Im}(f(z)) > 0$ $\forall z \in \mathbb{C}$. Pokaži, da je f konstanta

$g(z) = e^{if(z)}$, $f = u + iv$

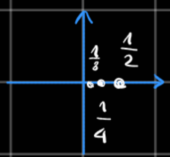
$g = e^{u - v}$

$|g(z)| = |e^{iu} e^{-v}| = e^{-v} \leq f \Rightarrow g = \text{konst.} \Rightarrow f = \text{konst.}$

④ Poišči hol. fjo $f: \mathbb{C} \to \mathbb{C}$, za katero velja $f\left(\frac{1}{n}\right) = e^{\frac{n^2 - 3n + 2}{n^3}}$

$n = 1, 2, 3, \ldots$

$$f\left(\frac{1}{n}\right) = e^{\frac{1}{n} - 3\left(\frac{1}{n}\right)^2 + 2\cdot\left(\frac{1}{n}\right)^3}$$



Naj bo $g: \mathbb{C} \to \mathbb{C}$; $g(z) = e^{z - 3z^2 + 2z^3}$ holom. ✓

$A = \left\{\frac{1}{n} ; n \in \mathbb{N}\right\} \Rightarrow f|_A = g|_A \Rightarrow A$ je množica s stekališčem 0

$\Rightarrow f = g$ na $\mathbb{C}$

Edina $f: \mathbb{C} \to \mathbb{C}$: $f\left(\frac{1}{n}\right) = e^{\frac{n^2 - 3n + 2}{n^3}}$ je $f(z) = e^{z - 3z^2 + 2z^3}$

$f: \mathbb{C} \to \mathbb{C}$ holo.

$f\left(\frac{1}{n}\right) = 0$ $\forall n$

$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$

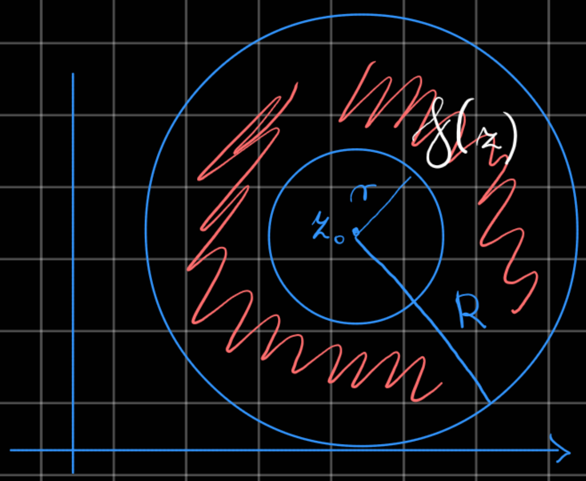
$\lim_{n \to \infty} f\left(\frac{1}{n}\right) = f(0) = c_0$

$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{n \to \infty} \frac{f\left(\frac{1}{n}\right) - 0}{\left(\frac{1}{n}\right)} = 0$

Laurentova vrsta

$r < |z - z_0| < R \quad f$ holom.



$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n =$$

$$= \dots + \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{(z - z_0)} + c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2$$

Residuum

Naj bo f holom. $0 < |z - z_0| < R$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

$$\operatorname{Res}(f, z_0) = c_{-1}$$

Če ima f v z_0 pol reda n

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left((z - z_0)^n f(z) \right)^{(n-1)}$$

Uporabnost

$f: D \to \mathbb{C}$ holom. fja, razen morda v $z_1, \dots z_k \in D$

$$\oint_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

5. Izračunaj $\operatorname{Res}\left(\frac{e^z}{\operatorname{sh}(z)}, 0\right)$

$$f(z) = \frac{e^z}{\operatorname{sh}(z)}, \quad z = 0 \Rightarrow \operatorname{sh}(z) = 0$$

$f(z)$ ima v $z=0$ pol 1. stopnje

L'Hôpital deluje v $\mathbb{C}$

$$\operatorname{Res}(f, z_0) = \lim_{z\to 0} \frac{1}{1}\left(z^1 \frac{e^z}{\operatorname{sh}(z)}\right) = \lim_{z\to 0} z \lim_{z\to 0} \frac{z}{\operatorname{sh}(z)}$$

$$= \lim_{z\to 0} \frac{1}{\operatorname{ch}(z)} = 1$$

6. $$\oint_{|z|=3} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i \left(\operatorname{Res}(f, 0) + \operatorname{Res}(f, 1)\right) = 2\pi i \left(1 - \frac{e}{2}\right)$$

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z \frac{e^z}{z(1-z)^3} = 1$$

pol 1. stopnje

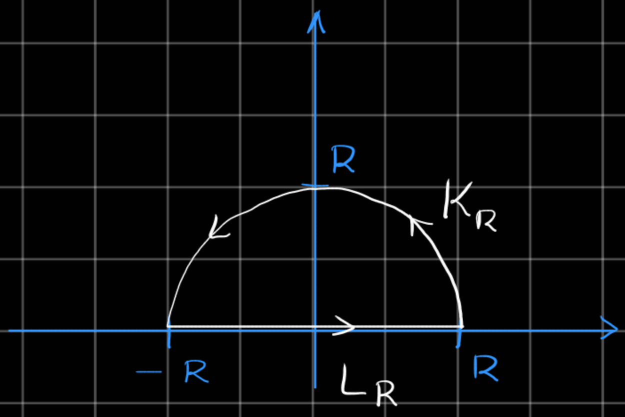
$$\operatorname{Res}(f, 1) = \lim_{z\to 1} \frac{1}{2!}\left((z-1)^3 \frac{e^z}{z(1-z)^3}\right)''$$

$$= \lim_{z\to 1}\left(-\frac{1}{2}\left(\frac{e^z}{z}\right)''\right) = -\frac{1}{2} e$$

Uporaba kompleksnih int. za izračun realnih integralov

i) $f: \mathbb{R} \to \mathbb{R}$ (brez polov v $\mathbb{R}$)

$$\int_{-\infty}^{\infty} f(x)\,dx \xrightarrow{\text{ideja}} F(z) \text{ holom., da } \underbrace{F(x) \text{"="} f(x)}_{\text{vrednosti se ujemajo}},\ x \in \mathbb{R}$$



$$\int_{L_R \cup K_R} F(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

$$\int_{-R}^{R} f(x)\,dx = \int_{K_R} F(z)\,dz$$

$$\downarrow R \to \infty \qquad \downarrow R \to \infty$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 0$$

7 Izračunaj $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$ (int. abs. konv.)

$$f(x) = \frac{\cos x}{1+x^2} \rightsquigarrow F(z) = \frac{e^{iz}}{1+z^2}$$

za $x \in \mathbb{R}$ se skoraj ujema $F(x) = \dfrac{\cos x + i \sin x}{1+x^2}$

$$\oint_{L_R \cup K_R} F(z)\,dz = \int_{L_R \cup K_R} \frac{e^{iz}}{1+z^2}\,dz \qquad // \text{ ni def., ko } 1+z^2 = 0 \Rightarrow z = \pm i$$

$$= 2\pi i \operatorname{Res}\Big(\frac{e^{iz}}{1+z^2}, \underset{\text{pol 1. st.}}{i}\Big) =$$

$$= 2\pi i \lim_{z\to i} (z-i)\frac{e^{iz}}{1+z^2} = 2\pi i \lim_{z\to i} \frac{e^{iz}}{(z+i)}$$

$$= 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

$$\int_{-R}^{R} F(z)\,dz + \int_{K_R} F(z)\,dz = \int_{-R}^{R} \frac{\cos x + i\sin x}{1+x^2}\,dx + \int_{K_R} F(z)\,dz$$

$$= \int_{-R}^{R} \frac{\cos x}{1+x^2} + i \underbrace{\int_{-R}^{R} \frac{\sin(x)}{1+x^2}}_{\substack{\text{liha fja} \\ \text{in je } 0\ \forall R}} + \int_{K_R} F(z)\,dz$$