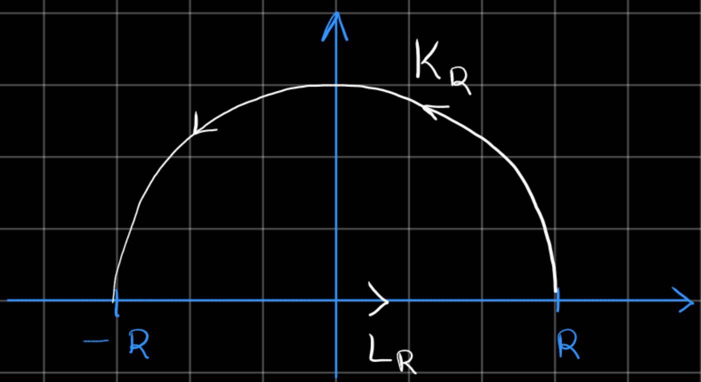
① Izračunaj $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx$

Rešitev (ideja = uporabi residuume)

$$f(z) = \frac{e^{iz}}{1+z^2} \qquad \left(x \in \mathbb{R},\ f(x) = \frac{\cos(x) + i\sin(x)}{1+x^2} \right)$$



// za residuum potrebujemo zanko

$$\oint_{L_R \cup K_R} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}(f, z_j) \qquad \text{// pol 1. stopnje}$$

$$= 2\pi i \operatorname{Res}(f(z), i) = 2\pi i \lim_{z\to i} (z-i) f(z) =$$

$$= 2\pi i \lim_{z\to i} \frac{(z-i)e^{iz}}{(z+i)(z-i)} = 2\pi i \frac{e^{iz}}{2i} = \frac{\pi}{e}$$

$$\frac{\pi}{e} = \oint_{L_R \cup K_R} f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{K_R} f(z)\,dz =$$

$L_R : z = x,\ x \in [-R, R]$

$dz = dx$

$$= \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx + \int_{K_R} f(z)\,dz =$$

$$= \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx + \underbrace{i\int_{-R}^{R} \frac{\sin x}{1+x^2}}_{} + \int_{K_R} f(z)\,dz =$$

liha fja po simetričnem integralu
$\Rightarrow \emptyset$

$$= \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx + \underbrace{\int_{K_R} f(z)\,dz}_{} \qquad (R \to \infty$$

kaj se zgodi s tem,
ko gre $R \to 0$

Ogledati si moramo

$$\left| \int_{K_R} f(z)\,dz \right| = \left| \int_0^{\pi} \frac{e^{iRe^{it}}\, Rie^{it}}{1+R^2e^{2it}}\,dt \right| \qquad \text{// ocenili bomo navzgor}$$

//parametrizacija K_R $\quad z = Re^{it},\ t \in [0, \pi]$

$$dz = Rie^{it}\,dt$$

$$\leq \int_0^{\pi} \left| \frac{e^{iR(\cos t + i\sin t)} Rie^{it}}{1+R^2e^{2it}} \right| dt = \int_0^{\pi} \frac{\overbrace{|e^{iR\cos t}}^{1} \cdot e^{-R\sin t}|\, R \cdot \overbrace{|i||e^{it}|}^{\text{oboje } 1}}{|1+R^2e^{2it}|}\,dt =$$

$$= \int_0^{\pi} \frac{e^{-R\sin t} \cdot R}{|1+R^2e^{2it}|}\,dt$$

R veliko število $\gg 1$, $e^{-R\sin t}$, eksponent je $\emptyset$ ali negativno

$$\leq \int_0^{\pi} \frac{R\,dt}{|1-R^2|} = \int_0^{\pi} \frac{R\,dt}{R^2-1} = \frac{R}{R^2-1}\cdot\pi$$

Uporabne neenakosti:

- $|z+w| \leq |z| + |w|$, $|z-w| \leq |z| + |w|$
- $|z-w| \geq ||z|-|w||$, $\boxed{|z+w| \geq ||z|-|w||}$ //to bi uporabimo

$$0 \leq \left| \int_{K_R} f(z)\,dz \right| \leq \frac{R}{R^2-1}\pi$$

$$\downarrow R\to\infty$$

$$0 \leq 0 \leq 0$$

$$\lim_{R\to\infty} \int_{K_R} f(z)\,dz = 0$$

$$\frac{\pi}{e} = \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx + \underbrace{\int_{K_R} f(z)\,dz}_{\to 0} \quad / \lim_{R\to\infty}$$

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$$

Integrali tipa (R)   // realni int.

$$\boxed{\int_0^{2\pi} f(\cos x, \sin x)\,dx = \oint_{S^1} f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}}$$

// nadomestimo $\cos x = \dfrac{z+\frac{1}{z}}{2}$, $\sin x = \dfrac{z-\frac{1}{z}}{2i}$, $dx = \dfrac{dz}{iz}$

↘ krožnica, enotska

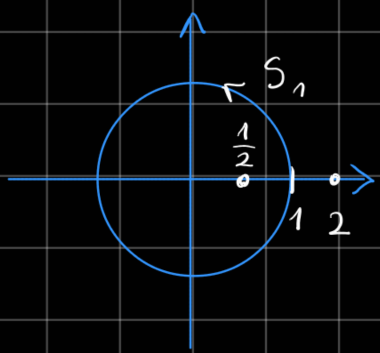
Slednji int. po krožnici lahko izračunamo z residuumi

② Izračunaj

$$\int_0^{2\pi} \frac{dx}{5-4x} = \oint_{S^1} \frac{\frac{dz}{iz}}{5-4\,\frac{z+\frac{1}{z}}{2}} = \oint_{S^1} \frac{\frac{dz}{iz}}{5-2\left(z+\frac{1}{z}\right)} =$$

$$= \oint_{S^1} \frac{\overbrace{\frac{1}{i}}^{=-i} dz}{5z-2z^2-2} = \oint_{S^1} \frac{i\,dz}{2z^2-5z+2} = \oint_{S^1} \frac{i\,dz}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$f(z) = \frac{i}{2\left(z-\frac{1}{2}\right)(z-2)}$$



// pol znotraj zanke je v 1/2

$$= 2\pi i \operatorname{Res}\left(\frac{i}{2\left(z-\frac{1}{2}\right)(z-2)}, \frac{1}{2}\right) = 2\pi i \lim_{z\to\frac{1}{2}} \frac{\cancel{\left(z-\frac{1}{2}\right)}\,i}{2\cancel{\left(z-\frac{1}{2}\right)}(z-2)} =$$

$$= 2\pi \cancel{x} \frac{\cancel{x}}{\cancel{2}\cdot(\cancel{x}\frac{3}{\cancel{2}})} = \frac{2\pi}{3}$$

③ Izračunaj

$$\int_0^\infty \frac{dx}{1+x^n}, \quad n = 2, 3, 4, \ldots$$ // v Mat3 z B fjo

$$B(a,b) = \int_0^\infty \frac{x^{a-1}dx}{(1+x)^{a+b}}$$

Op. Če je $n=1$, int. divergira

Poiščimo hol. fjo, ki je podobna

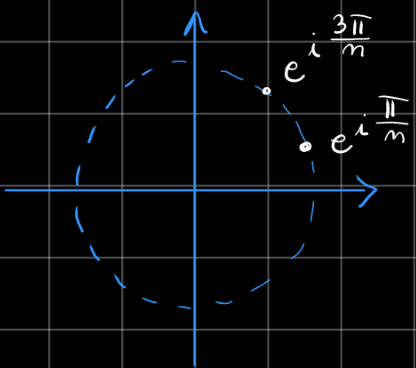
$$f(z) = \frac{1}{1+z^n} \quad \left(\text{hol. fja: } x \in \mathbb{R},\ f(x) = \frac{1}{1+x^n}\right)$$

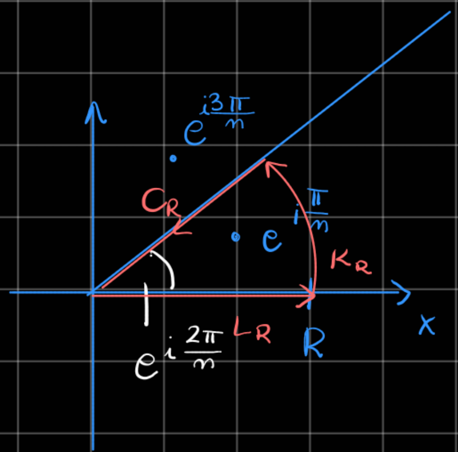
Poli fje: $z^n = -1$ (de Moivre)

$$z^n = e^{i\pi}, \quad k = 0, 1, 2, \ldots, n-1$$

$$z_k = e^{i\frac{\pi + 2k\pi}{n}}$$

k	z_k
0	$e^{i\frac{\pi}{n}}$
1	$e^{i\frac{3\pi}{n}}$
2	





R>>1, ničla leži v zanki

$$\oint_{L_R \cup K_R \cup C_R} \frac{dz}{1+z^n} = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^n}, \underbrace{e^{i\frac{\pi}{n}}}_{\text{pol 1. stopnje}}\right) =$$

$$= 2\pi i \lim_{z \to e^{i\frac{\pi}{n}}} \frac{(z - e^{i\frac{\pi}{n}})\, 1}{1 + z^n} \underset{\text{L'Hop}}{=} 2\pi i \lim_{z \to e^{i\frac{\pi}{n}}} \frac{1}{n z^{n-1}} = \frac{2\pi i}{n e^{i\frac{(n-1)\pi}{n}}}$$

// Rezultat integrala po zanki

$$\oint_{L_R \cup K_R \cup C_R} f(z)\, dz = \int_{L_R} \frac{dz}{1+z^n} + \int_{K_R} f(z)\, dz + \int_{C_R} \frac{dz}{1+z^n}$$

obrnemo, da integriramo $-C_R$

$$\int_{L_R} \frac{dz}{1+z^n} : \quad z = x, \quad x \in [0, R], \quad dz = dx$$

$$\int_{C_R} \frac{dz}{1+z^n} : \quad z = x e^{i\frac{2\pi}{n}}, \quad x \in [0, R]$$

naklon premice

$$dz = dx\, e^{i\frac{2\pi}{n}}$$

$$= \int_0^R \frac{dx}{1+x^n} - \int_0^R \frac{e^{i\frac{2\pi}{n}}\, dx}{1 + (x \underbrace{e^{i\frac{2\pi}{\cancel{n}}}}_{})^{\cancel{n}}} + \int_{K_R} f(z)\, dz =$$

$$= \int_0^R \frac{dx}{1+x^n} - e^{i\frac{2\pi}{n}} \int_0^{R} \frac{dx}{1+x^n} + \int_{K_R} f(z)\, dz =$$

$$= \left(1 - e^{i\frac{2\pi}{n}}\right) \underbrace{\int_0^R \frac{dx}{1+x^n}}_{\text{ko gre } R \to \infty \text{, je naš integral}} + \overbrace{\int_{K_R} f(z)\, dz}^{\text{koliko je pa vrednost tega?}}$$

$$\left|\int_{K_R}\frac{dz}{1+z^n}\right| = \left|\int_0^{\frac{2\pi}{n}}\frac{Rie^{it}dt}{1+R^n e^{int}}\right| \leq \int_0^{\frac{2\pi}{n}}\frac{|R\overbrace{ie^{it}}^{1}|dt}{|1+R^n\underbrace{e^{int}}_{1}|} \leq \int_0^{\frac{2\pi}{n}}\frac{R\,dt}{R^n-1}$$

// za $R \gg 1$

$K_R: z = Re^{it}, \quad t\in[0,\frac{2\pi}{n}]$

$$dz = Rie^{it}dt$$

$$= \frac{R}{R^n-1}\cdot\frac{2\pi}{n} \xrightarrow{R\to\infty} \emptyset$$

$$\Rightarrow \int_{K_R} f(z)dz \longrightarrow 0 \quad \text{za } R\to\infty$$

$$\frac{2\pi i}{n e^{i\frac{(n-1)\pi}{n}}} = \left(1-e^{i\frac{2\pi}{n}}\right)\int_0^R\frac{dx}{1+x^n} + \int_{K_R} f(z)dz, \quad R\to\infty$$

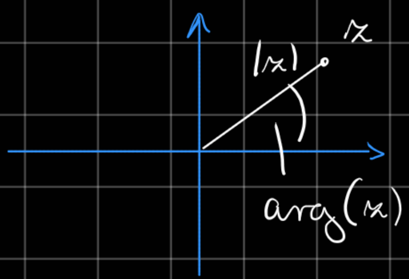
$$\int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi i}{n e^{i\frac{(n-1)\pi}{n}}\left(1-e^{i\frac{2\pi}{n}}\right)} = \frac{2\pi i}{n e^{i\pi} e^{-i\frac{\pi}{n}}\left(1-e^{i\frac{2\pi}{n}}\right)}$$

$$= -\frac{2\pi i\, e^{i\frac{\pi}{n}}}{n\left(1-e^{i\frac{2\pi}{n}}\right)} = \ldots = \frac{\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

Naravni logaritem

$$\ln(z) = \ln|z| + i\arg(z)$$



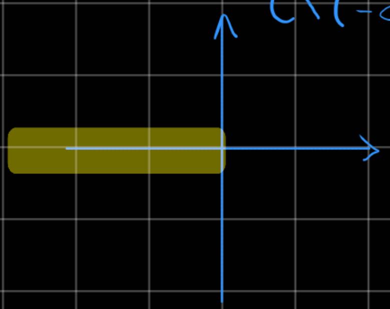
Def. območje, $\ln(z)$ ni definiran v $z = 0$

$\ln(z)$ ni zvezna na $\mathbb{C} \setminus \{0\}$

Želimo def. območje, kjer je $\ln(z) \Rightarrow$ holomorfna

Npr.

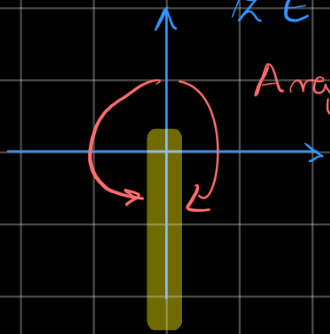
$\mathbb{C} \setminus (-\infty, 0)$



,

$z \in \mathbb{C} \setminus i(-\infty, 0]$

$\mathrm{Arg}(z) \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$

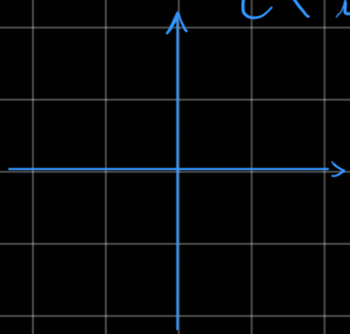


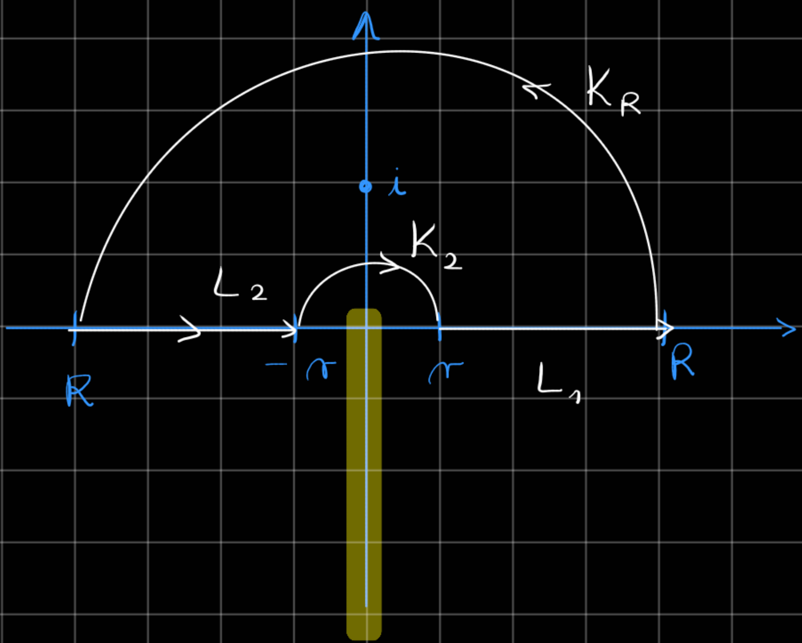
Za zveznost odstranimo poltrak

(4) Izračunaj $\displaystyle\int_0^\infty \frac{\ln(x)}{1+x^2}\,dx$

$f(z) = \dfrac{\ln(z)}{1+z^2} \longrightarrow$ D_f za $\ln(z)$

$\mathbb{C} \setminus i(-\infty, 0]$





$(R \gg 1,\ r \approx 0)$

$$\oint_{L_1 \cup K_R \cup L_2 \cup K_r} \frac{\ln(z)}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left(\frac{\ln z}{1+z^2}, i\right) =$$

$$= 2\pi i \lim_{z \to i} \frac{(z-i)\ln(z)}{(z+i)(z-i)} = 2\pi i \frac{\ln(i)}{2i}$$

$$= \pi\left(\underbrace{\ln \underbrace{|i|}_{1}}_{\varnothing} + i\underbrace{\arg(i)}_{\frac{\pi}{2}}\right) = \frac{\pi^2}{2} i$$

$$\oint_{L_1 \cup K_R \cup L_2 \cup K_r} \frac{\ln(z)}{1+z^2}\,dz = \int_{L_1} \frac{\ln(z)}{1+z^2}\,dz + \int_{K_R} \frac{\ln(z)}{1+z^2}\,dz \quad \int_{L_2} \frac{\ln(z)}{1+z^2}\,dz$$

$$+ \int_{K_r} \frac{\ln(z)}{1+z^2}\,dz =$$

$L_1 : x \in [r, R]$

$L_2 : x \in [-R, -r] \qquad z = -x \quad dz = -dx$

$$= \int_r^R \frac{\ln(x)}{1+x^2}\,dx + \int_{K_R} \frac{\ln(z)}{1+z^2}\,dz + \int_{K_r} \frac{\ln(z)}{1+z^2}\,dz - \int_{-L_2} \frac{\ln(z)}{1+z^2}\,dz$$

$$= \int_r^R \frac{\ln x}{1+x^2}\,dx + \int_{K_R} \frac{\ln(z)}{1+z^2}\,dz + \int_{K_r} \frac{\ln(z)}{1+z^2} + \underbrace{\int_r^R \frac{\ln(-x)\,dx}{1+x^2}}$$

// definicija logaritma $\quad \ln|-x| + \pi i$

$$= \int_r^R \frac{\ln x}{1+x^2}\,dx + \int_r^R \frac{\ln x + i\pi}{1+x^2}\,dx + \int_{K_R} + \int_{K_r}$$

$$= 2\int_r^R \frac{\ln(x)}{1+x^2}\,dx + i\pi \int_r^R \frac{dx}{1+x^2} + \int_{K_R} + \int_{K_r}$$

$$= 2\int_r^R \frac{\ln x}{1+x^2} + i\pi\left(\operatorname{arctg}(R) - \operatorname{arctg}(r)\right) + \int_{K_R} + \int_{K_r}$$

Uvemo, kaj se dogaja, ko gresta $r \to 0$ in $R \to \infty$ pri prvih dveh int.

$$\left|\int_{K_R} \frac{\ln x}{1+x^2}\,dx\right| = \left|\int_0^\pi \frac{\overbrace{\ln(Re^{it})}^{\to \ln R + it} Rie^{it}\,dt}{1+R^2e^{i2t}}\right| \leq$$

$x = R\,e^{it}$

$$\int_0^\pi \frac{R\,|\ln R + it|}{|1+R^2e^{i2t}|} \leq$$

$dx = Rie^{it}dt$

za vrednost t

$$\leq \int_0^\pi \frac{R(\ln R + \pi)}{R^2-1}\,dt = \underbrace{\frac{\pi R(\ln R+\pi)}{R^2-1}}$$

$\downarrow R \to \infty$

$\emptyset$

$$\left|\int_{K_r} \frac{\ln x}{1+x^2}\,dx\right| = \left|\int_0^\pi \frac{\ln(re^{it})\,rie^{it}\,dt}{1+r^2e^{it2}}\right| \leq$$

$x = r\,e^{it}$, $t \in [0,\pi]$

dx

$$\leq \int_0^{\pi} \frac{r\,|\ln r + i\pi|}{1-r^2}\,dt = \frac{\pi (\underbrace{r\,|\ln r|}_{\to 0} + \pi)}{\underbrace{1 - r^2}_{\to 1}} \xrightarrow{r \to 0} 0$$

$$\cancel{i\frac{\pi^2}{2}} = 2\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx + i\pi\,\cancel{(\operatorname{arctg}(\infty)} - \underbrace{\operatorname{arctg}(0))}_{=\frac{\pi}{2}}$$

$$\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx = 0$$

Biholomorfne preslikave

U, V območji v $\mathbb{C}$

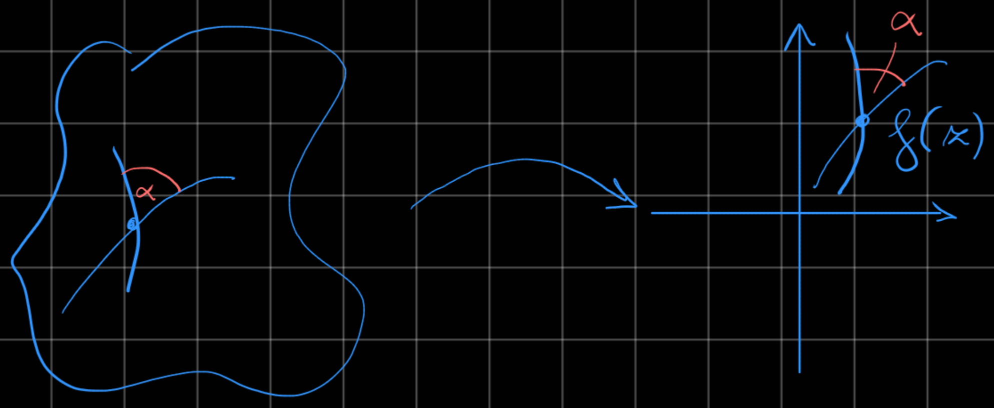
$f: U \to V$ je biholomorfna, či je f holomorfna, bijektivna in f^{-1} je holomorfna

(f bihol. slika obm. U na obm. V)

Op.: Če je f holomorfna & bijektivna $\Rightarrow f'(z) \neq 0 \Rightarrow f^{-1}$ holomorfna.

Velja:

1) Če je $f: U \to V$ holomorfna in $f'(z) \neq 0 \Rightarrow f$ konfo= rmna v



② Riemannov izrek: naj bo D odprta neprazna in enostavno povezana množica v $\mathbb{C}$, $D \neq \mathbb{C}$

$\Rightarrow \exists$ biholomorfna $f: D \longrightarrow B(0,1)$ // enotski disk

Op.: Za vaje ta izrek ni toliko uporaben