Biholomorfne preslikave

D_1, D_2 območji v $\mathbb{C}$

$f: D_1 \to D_2$ je biholomorfna, če je f holomorfna, bijektivna in f^{-1} holomorfna.

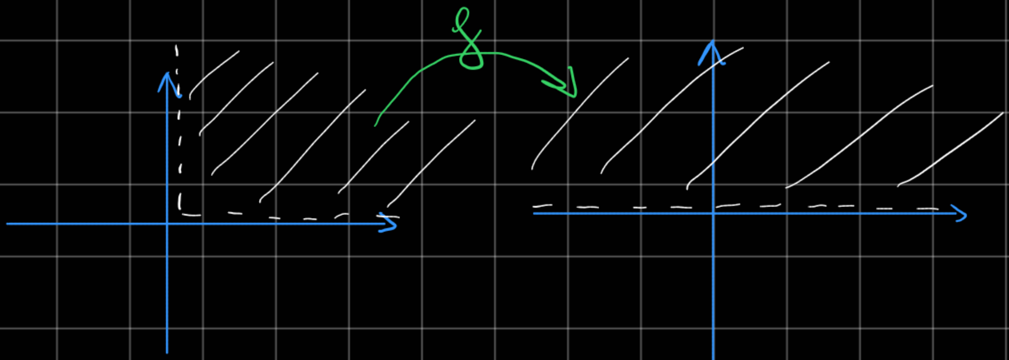
f holomorfna in bijektivna $\Rightarrow$ biholomorfna

Tipična naloga: podani območji D_1, D_2 $\Rightarrow$ poišči f biholofomno $D_1 \to D_2$

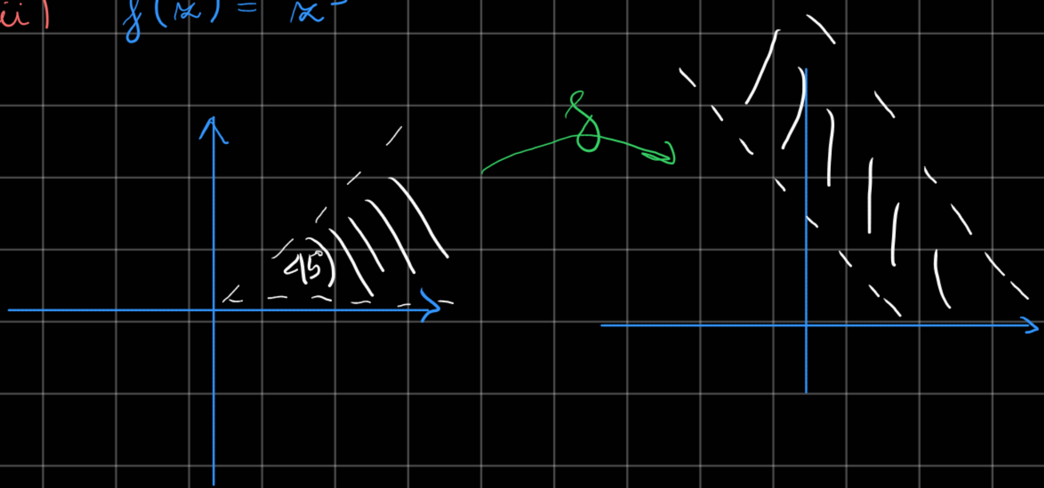
Riemannov izrek: D enostavno pov. obm. v $\mathbb{C}$, $D \neq \mathbb{C} \Rightarrow \exists$ BIHOL.

$f: D \to B(0,1)$

Primeri: i) $f(z) = z^2$ $\qquad f(re^{i\varphi}) = r^2 e^{i2\varphi}$



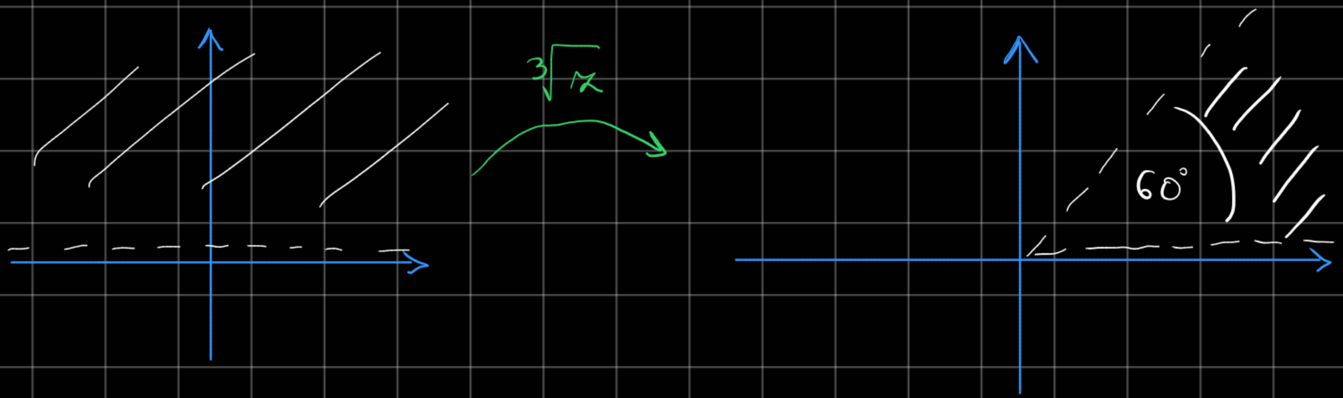
ii) $f(z) = z^3$



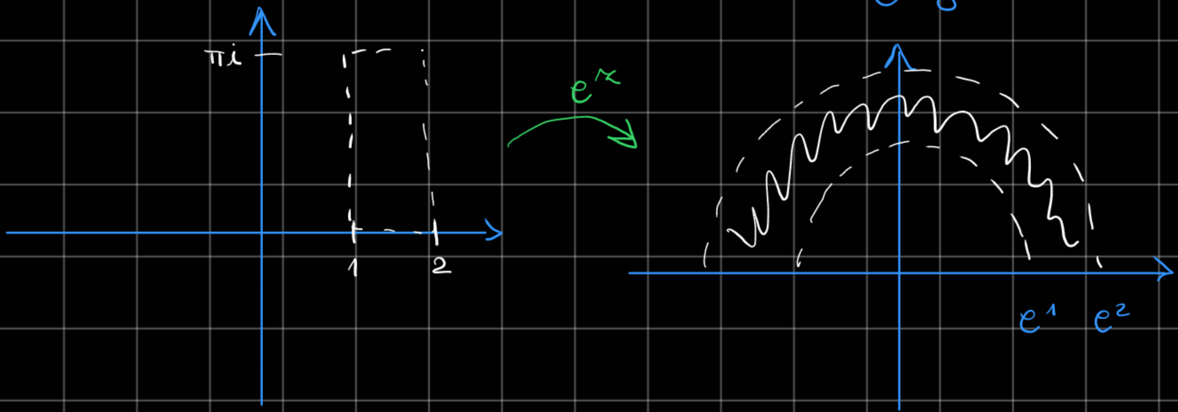
iii) $f(z) = \sqrt[3]{z}$

$$f(re^{i\varphi}) = r^{\frac{1}{3}} e^{i\frac{\varphi}{3}} \qquad \left(\sqrt[3]{z}\right)^3 = z$$

// 3 moguće orijentacije



iv) $f(z) = e^z \Rightarrow f(x+iy) = e^x \underbrace{(\cos y + i \sin y)}_{e^{iy}}$



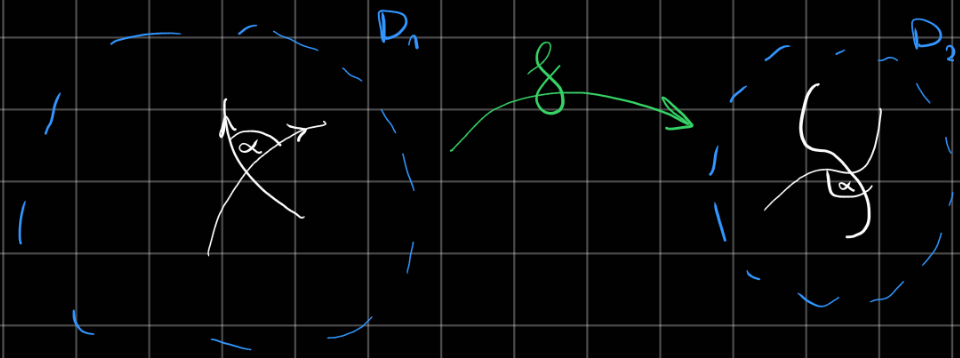
Möbiusove transformacije

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C} \qquad (f: \mathbb{C} \to \mathbb{C})$$

ni def. $z = -\frac{d}{c}$

Svojstva

i) f je konformna preslikava (kot vsaka biholomorfna)



ii) Möb transf. slikajo krožnica & premice $\longrightarrow$ krožnica & premica

premica = krožnica z ∞ radijem

iii) Möb. transf. $f(z) = \dfrac{az+b}{cz+d}$ je enolično določena s sliko 3 točk

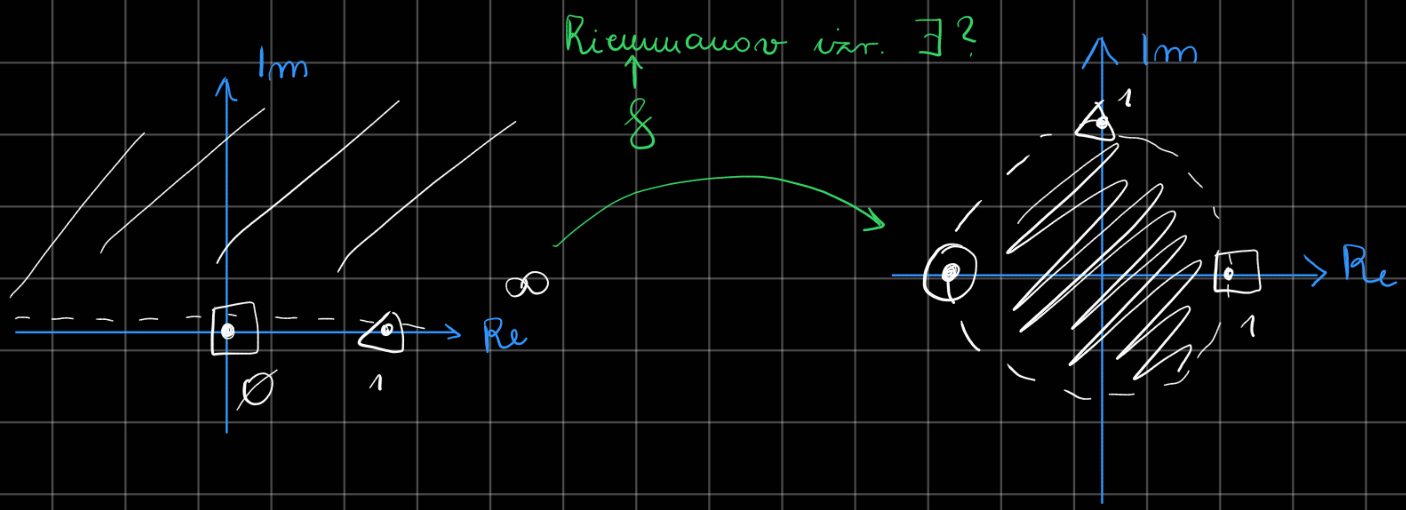
P: $z_1, z_2, z_3 \in \mathbb{C}$

$$f(z) = \frac{z - z_1}{z - z_3} \, \frac{z_2 - z_3}{z_2 - z_1} \Rightarrow \begin{aligned} f(z_1) &= 0 \\ f(z_2) &= 1 \\ f(z_3) &= \infty \end{aligned}$$

① Poišči bihol. preslikavo

$$f: \{z \in \mathbb{C} \,;\, \mathrm{Im}(z) > 0\} \longrightarrow B(0,1) = \{z \in \mathbb{C},\ |z| < 1\}$$

Namig: poišči Möb. transf. (na ustrezen način je treba slikati 3 točke)



Iščemo Möb transf.

$f(0) = 1 \quad \Rightarrow f(\emptyset) = 1 = \frac{a \cdot \emptyset + b}{c\emptyset + d} = 1 = \frac{b}{d} \Rightarrow b = d$

$f(1) = i$

$f(\infty) = -1 \quad \Rightarrow f(x) = \frac{ax + b \,:x}{cx + d \,:x} \Rightarrow f(\infty) = \frac{a\infty + b}{c\infty + d} \approx \frac{a\infty}{c\infty}$

$\Rightarrow -1 = \frac{a}{c}$

$-a = c$

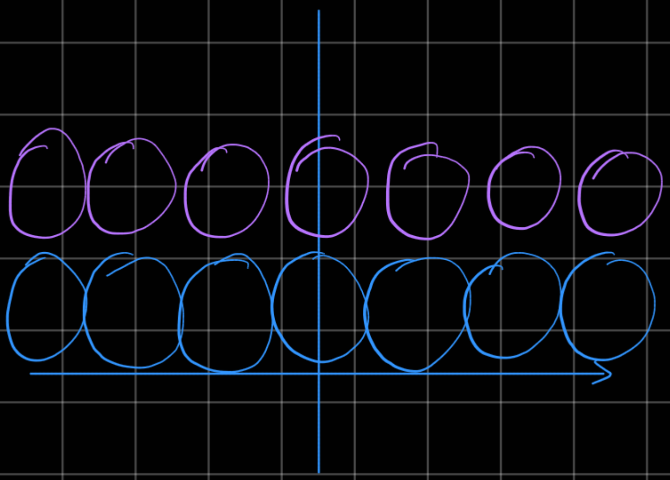
$f(x) = \frac{az + b}{-az + b} \Rightarrow f(1) = \frac{a+b}{b-a} = i \Rightarrow ai + bi = b - a$

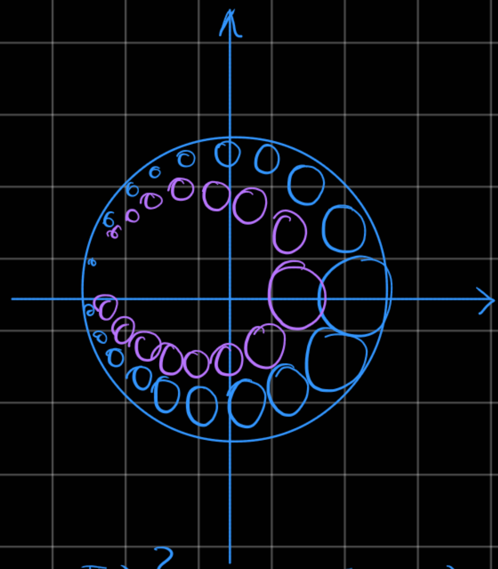
$a(-1-i) = a(1-i)$

$\Rightarrow b = a\frac{1+i}{1-i} = -a\frac{(1+i)(1+i)}{(1-i)(1+i)} = a\frac{(1+i)^2}{2} = -ai$

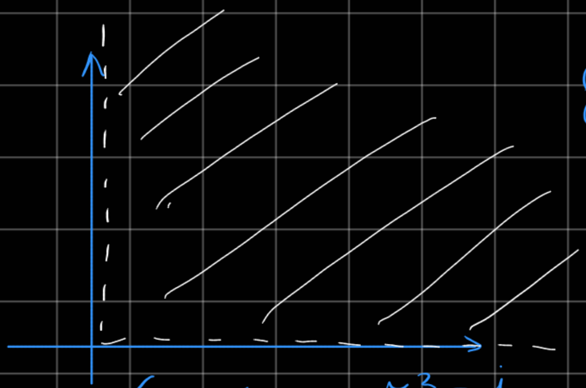
$f(x) = \frac{az - ai}{-az - ai}$

f je konformna & ohranja preslikave

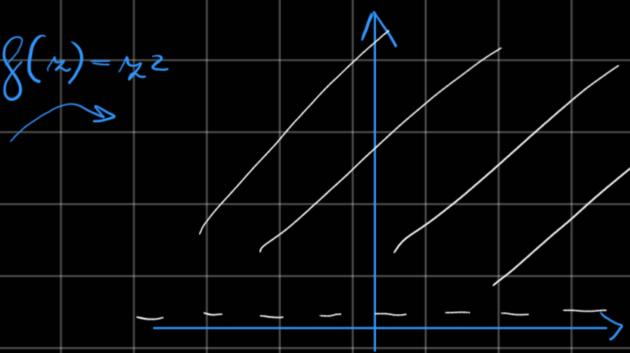




② Poișcă bihol. $f : \{z \in \mathbb{C},\ Arg(z) \in (0, \frac{\pi}{2})\} \to B(0,1)$

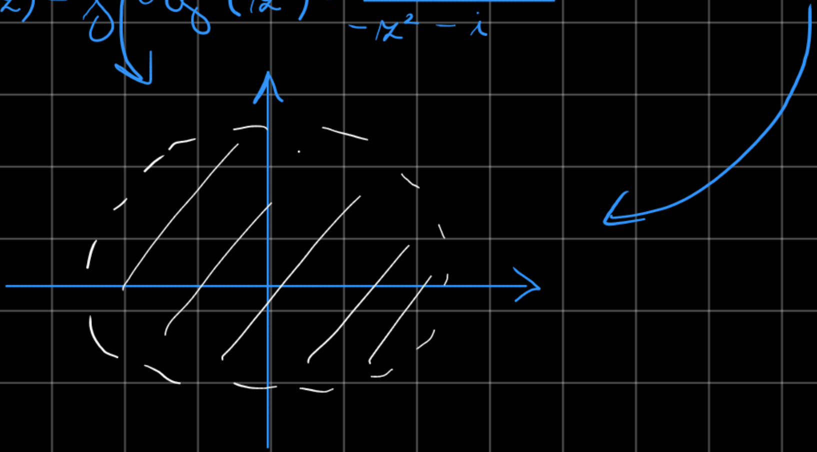


$f(z) = z^2$



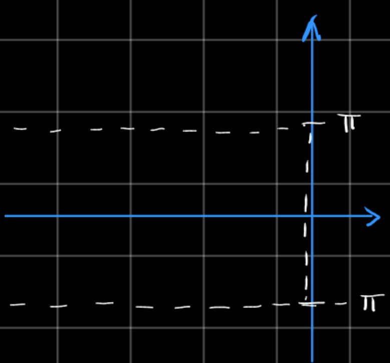
$F(z) = f \circ g\,(z) = \dfrac{z^2 - i}{-z^2 - i}$

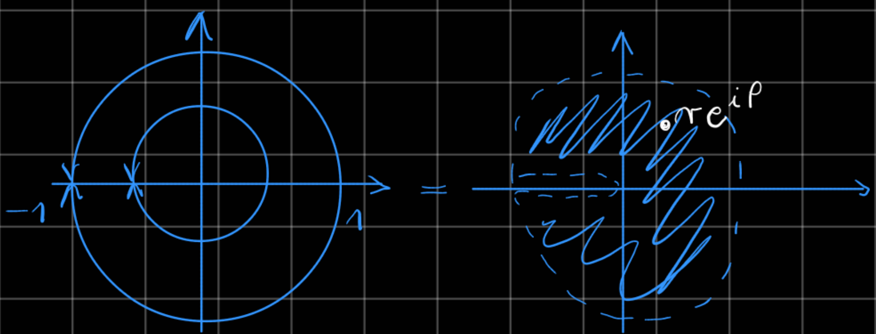
$f(z) = \dfrac{z - i}{-z - i}$



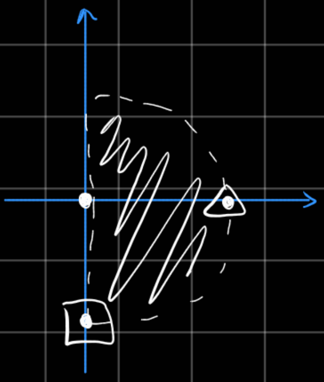
③ Poișcă BIHOL

$f : \{z \in \mathbb{C},\ Re(z) < 0 \text{ și } Im(z) \in (-\pi, \pi)\}$

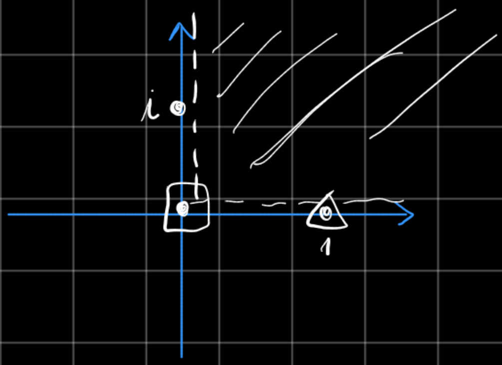




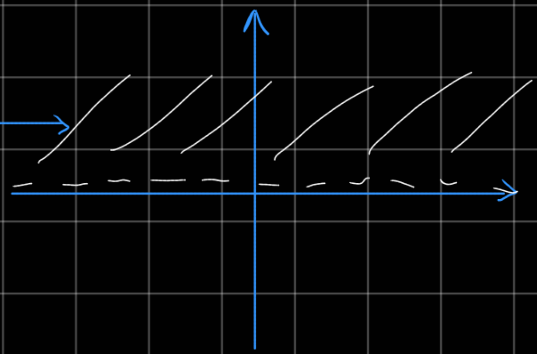
$\sqrt{z}$



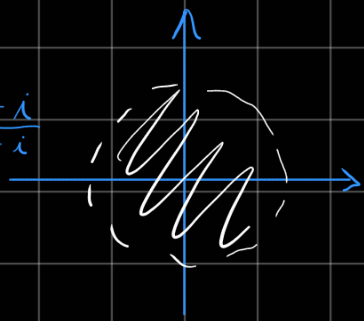
$f(z) = \dfrac{z-i}{-z-i}$



z^2



$g(z) = \dfrac{z-i}{-z-i}$



$f(-i) = 0$

$f(i) = \infty \quad \Rightarrow \quad d = -ic$

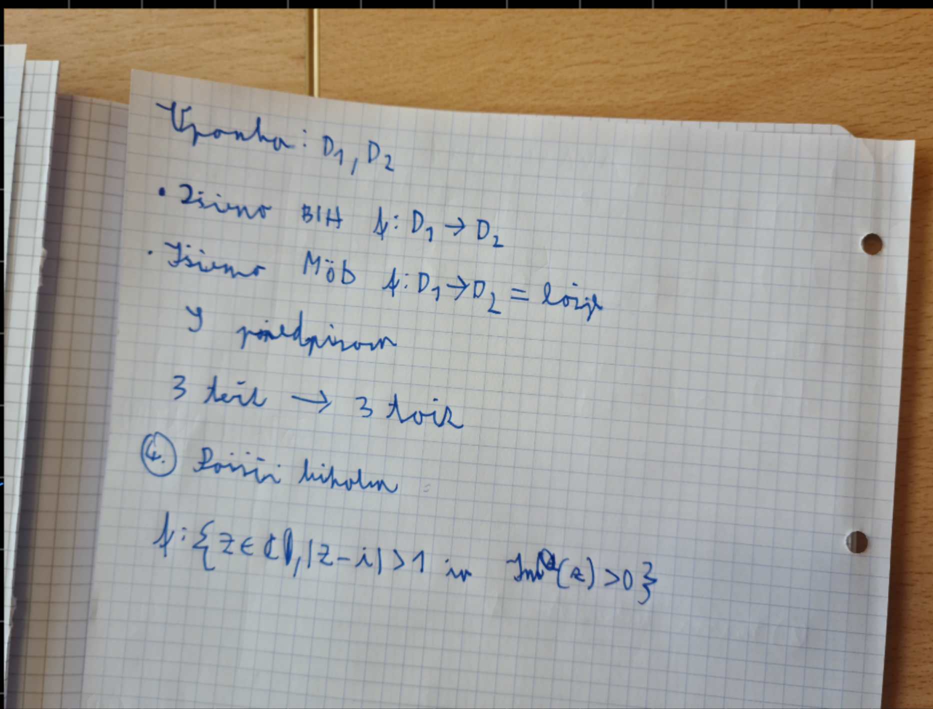
$f(1) = 1$

$c - ci = a + ai \quad \Rightarrow \quad c = ai$

$f(z) = \dfrac{z+i}{iz+1}$

Op.: D_1, D_2

- Iščemo bihol. $f: D_1 \to D_2$

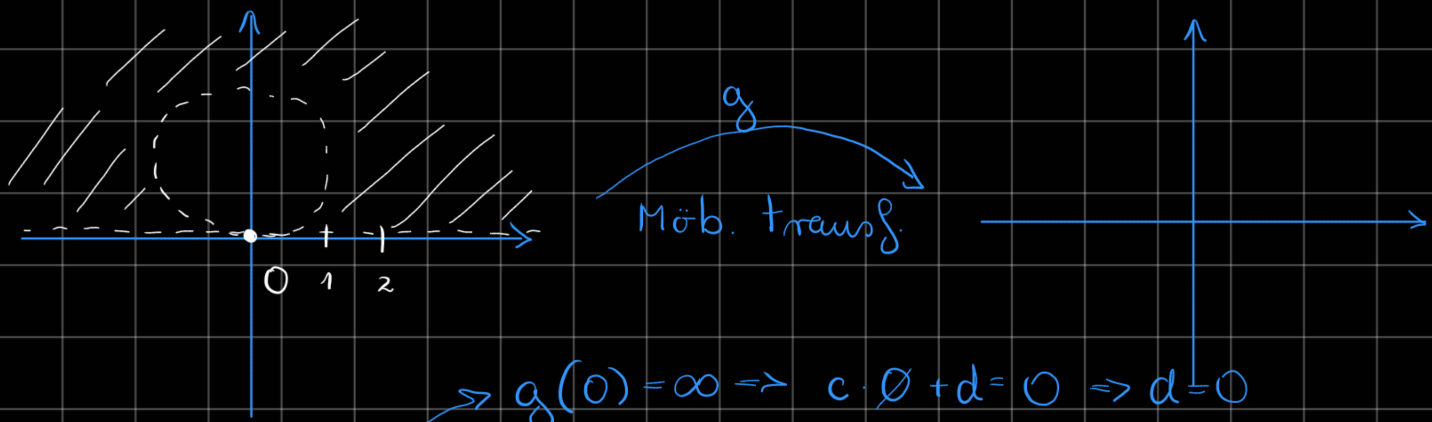


④ Poišči bihol.

$$g: \{ z \in \mathbb{C},\ |z-i| > 1 \text{ in } \operatorname{Im}(z) > 0 \}$$

$$\downarrow$$

$$\{ z \in \mathbb{C};\ \operatorname{Im}(z) \in (-\pi, \pi) \}$$



$$g(0) = \infty \Rightarrow c \cdot 0 + d = 0 \Rightarrow d = 0$$

$$\Downarrow$$

$$g(z) = \frac{az + b}{cz}$$

$g(0) = \infty$

$g(1) = 0$

$g(2) = 1$

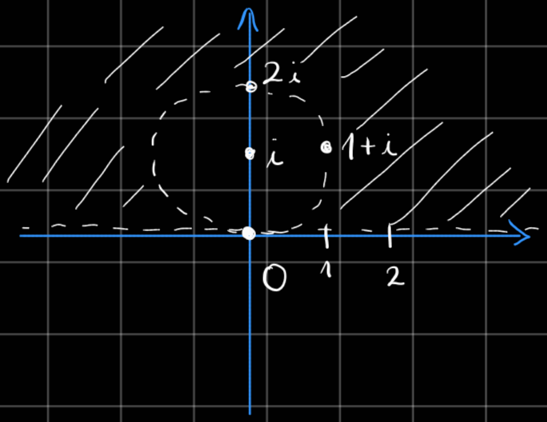
$$g(1) = \frac{a+b}{c} = 0 \Rightarrow b = -a$$

$$g(z) = \frac{az - a}{cz}$$

$$\frac{2a - a}{2c} = 1 \Rightarrow a = 2c \Rightarrow c = \frac{a}{2}$$

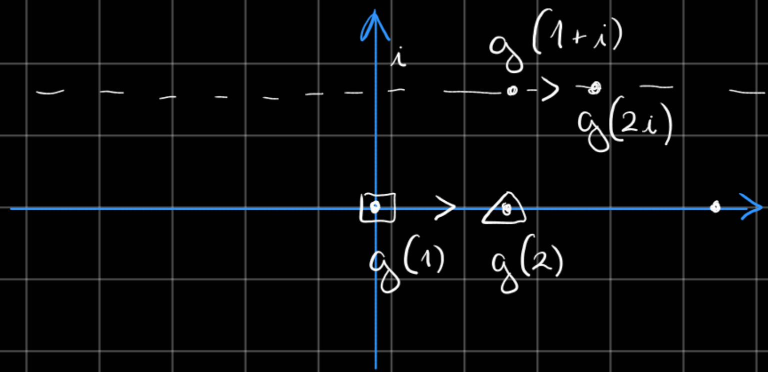
$$\Rightarrow g(z) = \frac{2(z-1)}{z}$$

Sedaj preverimo če



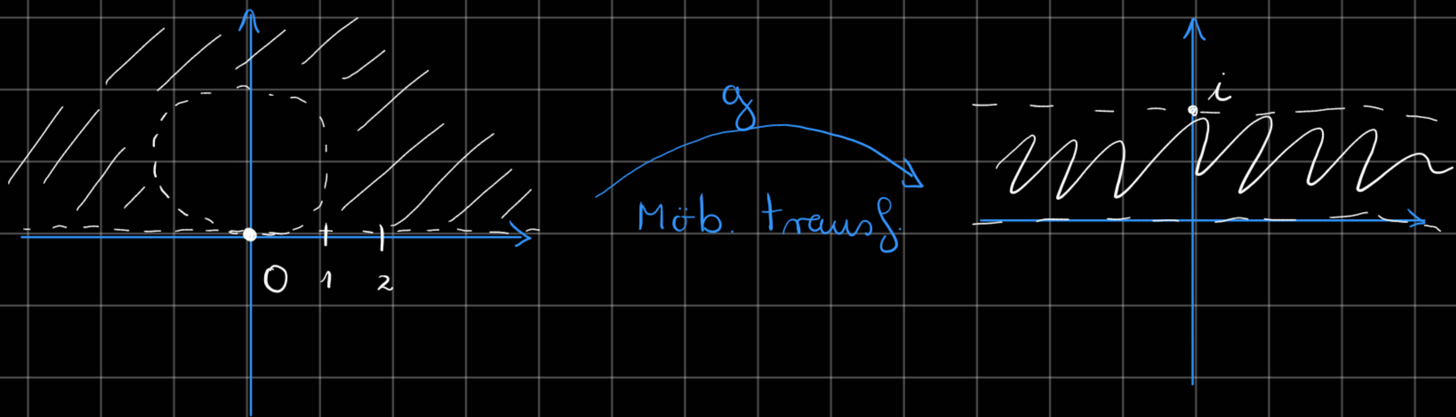
$$g(2i) = \frac{4i-2}{2i} = 2+i$$

$$g(1+i) = 1+i$$



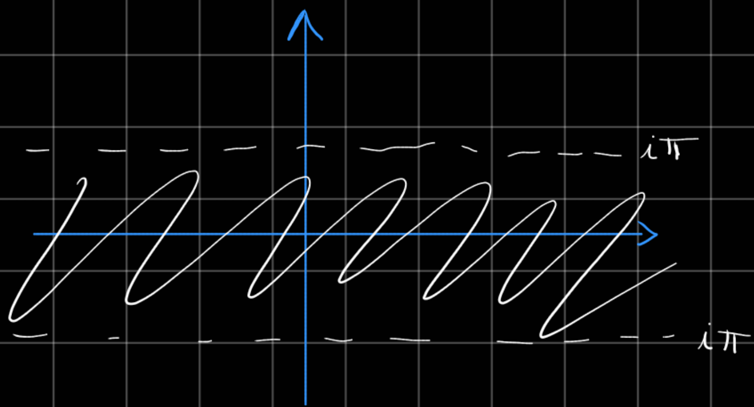
$x \in \mathbb{R}$

$$\lim_{x \downarrow 0} g(x) = \lim_{x \downarrow 0} \frac{2x-2}{x} = -\infty$$



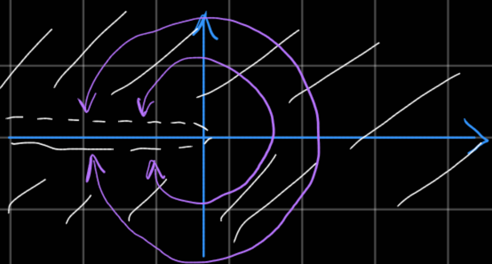
$\xrightarrow{h}$

$h(z) = 2\pi z \quad i\pi$

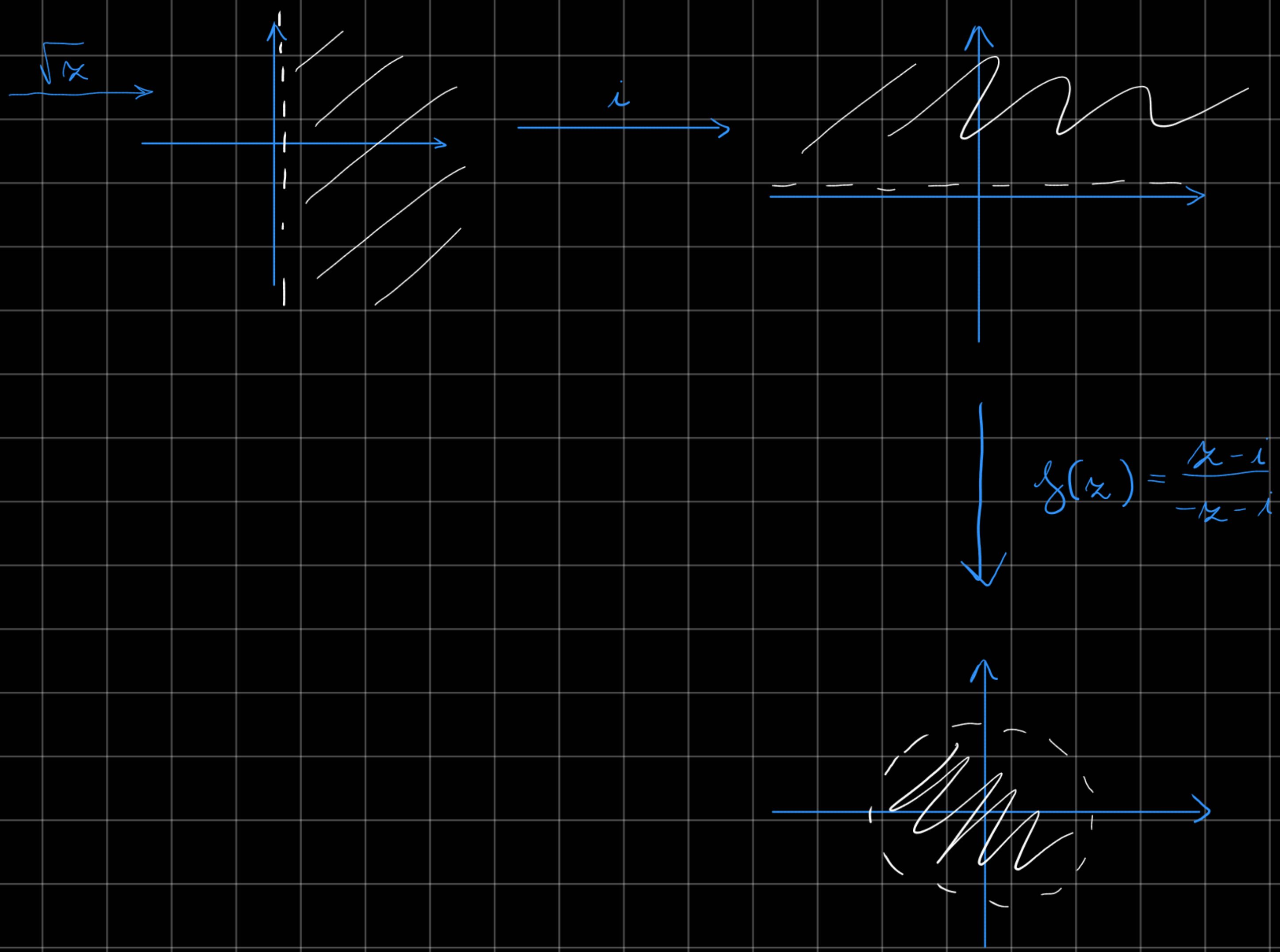


Op. Nadaljevanje naloge na $B(0,1)$

$\xrightarrow{e^z}$



// ker so vse realne vrednosti, dobimo koncentrične kroge glej ③


$\sqrt{z}$
i
$g(z) = \frac{z-i}{-z-i}$