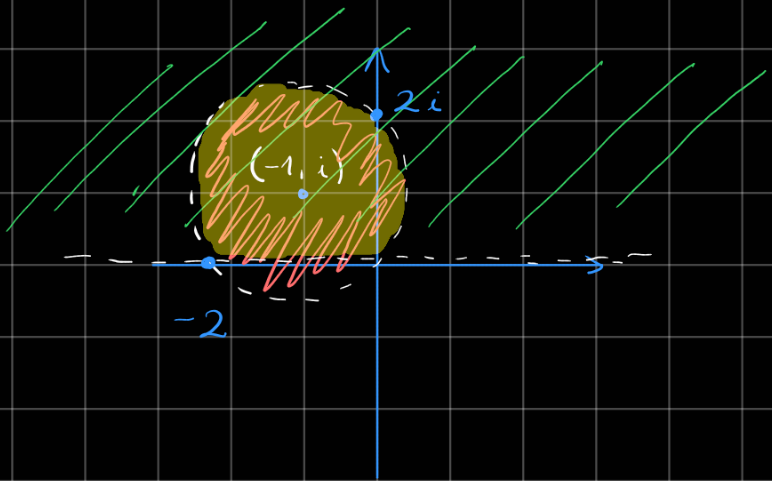
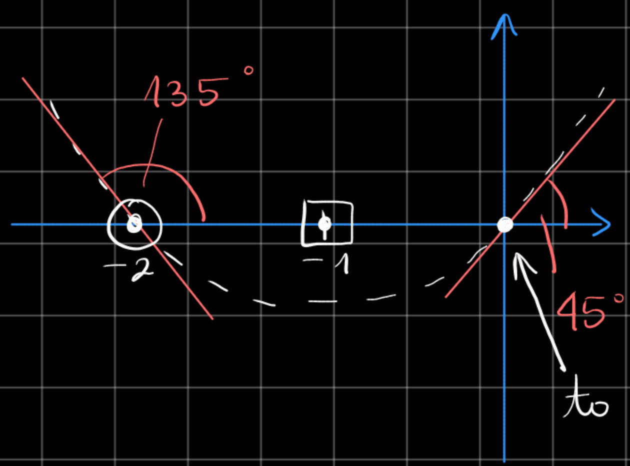
Zadnja naloga od zadnjič

① Poišči bihol. preslikavo

$$\{z \in \mathbb{C},\ |z+1-i| < \sqrt{2} \text{ in } \operatorname{Im}(z) > 0\} \longrightarrow B(0,1)$$





Direktno z Möbiusovo transf. ne gre, saj bi rabili ohranjati kote

to točko slikamo v ∞

$f(0) = \infty$

$f(-2) = 0$

$f(-1) = 1$

Kar naredimo je, da odpremo pri točki $(0,0)$

$$f(z) = \frac{az+b}{cz+d}$$

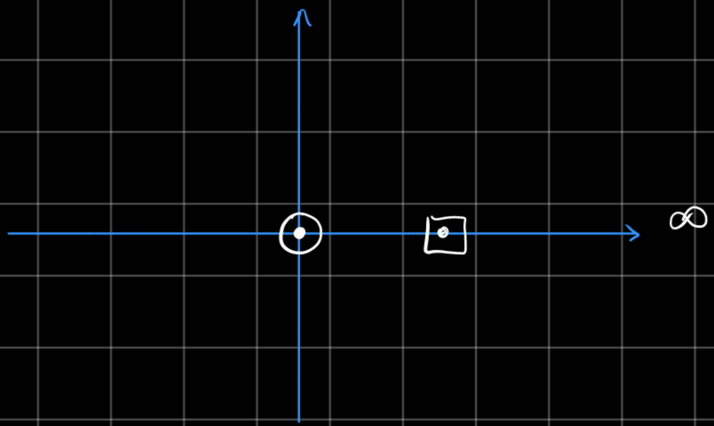
1) $f(0) = \frac{b}{d} = \infty \Rightarrow d = \emptyset$

2) $f(-2) = 0 \Rightarrow -2a + b = 0 \Rightarrow b = +2a$

$$f(z) = \frac{az + 2a}{cz}$$

3) $f(-1) = 1$ $\quad 1 = \frac{-a + 2a}{-c} \Rightarrow -c = a$

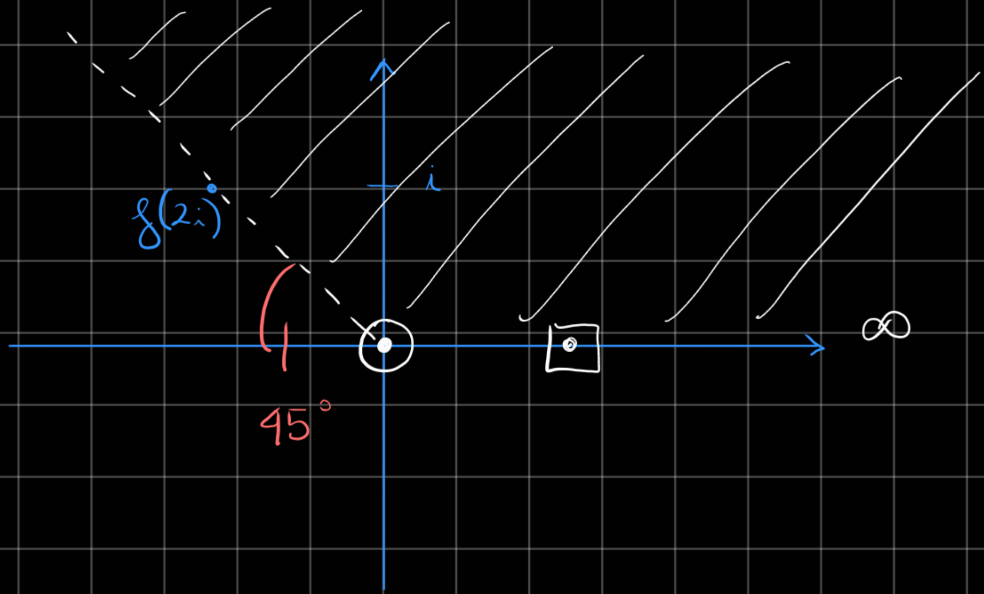
$$\boxed{f(z) = \frac{-z-2}{z}}$$



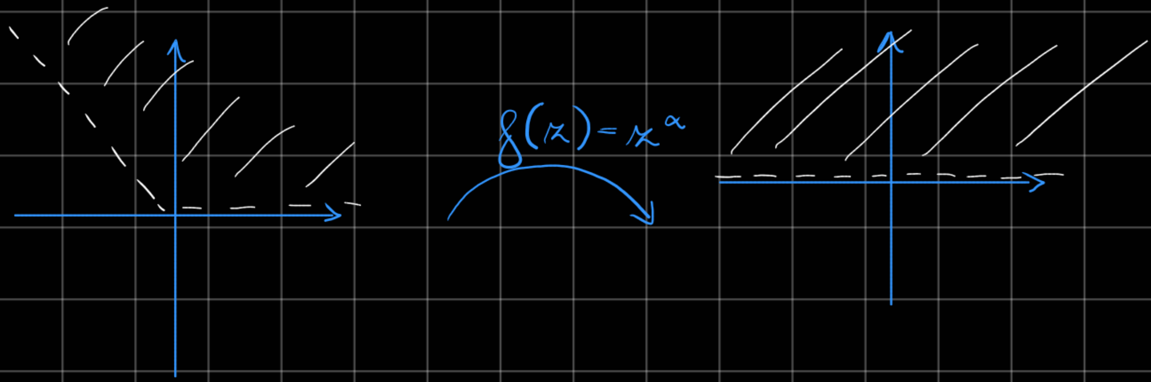
// slika območja

Poglejmo, kam se slika $2i$

$$f(2i) = \frac{-2i-2}{2i} = -1 + i$$



Najti moramo preslikavo, da preslika na zg. polravnino



Preslikava z^α naj bo dana z

$z = re^{i\varphi} \Rightarrow z^\alpha = r^\alpha e^{i\alpha\varphi}$ // splošna fja

$z = re^{i\phi} \Rightarrow z^\alpha = r^\alpha e^{i\phi}$

$z = re^{i\frac{3\pi}{4}} \Rightarrow z^\alpha = r^\alpha e^{i\alpha\frac{3\pi}{4}} = r^\alpha e^{i\pi} \Rightarrow \alpha = \frac{4}{3}$

$\alpha \frac{3\cancel{\pi}}{4} = \cancel{\pi}$

$$F(z) = \frac{\left(\frac{-z-2}{z}\right)^{\frac{4}{3}} - i}{-\left(\frac{z-2}{z}\right) - i}$$

// polravnina se slika š v krog

Harmonične fje in Greenova funkcija

$D^{odp} \subseteq \mathbb{R}^2$, $u: D \to \mathbb{R}$, $u(x,y)$

u je harmonična funkcija na D, č

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Pogosta naloga: Poišči fjo $u : \overline{D} \to \mathbb{R}$, da ji u harm.

na D in $u\big|_{\partial D} = f$

Velik vir harm. fj, so realni in imag. deli holom. fj. $\Rightarrow$ f holomorfnih f.

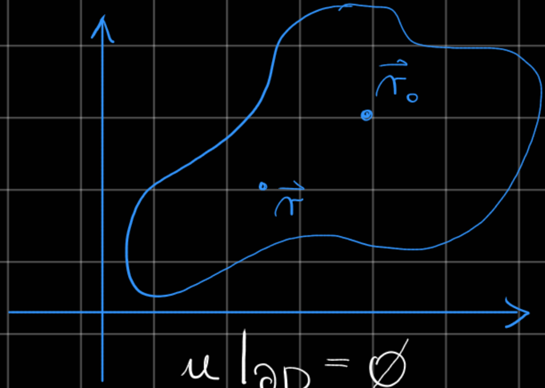
$\mathrm{Re}(f)$ = harmonična $\Rightarrow$ ima pa neke vrednosti na ∂D

Greenova fja območja D

$G(\vec{r}, \vec{r}_0)$, $G : \overline{D} \times D \longrightarrow \mathbb{R}$

i) $G(\vec{r}, \vec{r}_0) = 0 \quad \forall \vec{r} \in \partial D$

ii) $\Delta_{\vec{r}}\, G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$



$u\big|_{\partial D} = \emptyset$

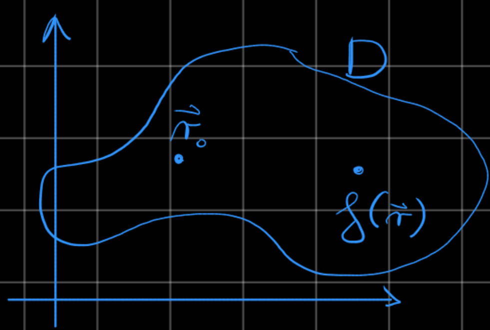
$\Delta u = \delta(\vec{r} - \vec{r}_0)$

Zelo hitro lahko zapišemo z G rešitev še enega problema.

$u : \overline{D} \to \mathbb{R}$

$u\big|_{\partial D} = 0$ (BC)

$\Delta u = f(\vec{r})$



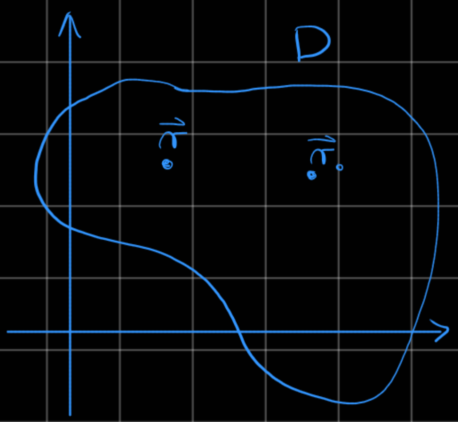
$$u(\vec{r}_0) = \iint_D f(\vec{r})\, G(\vec{r}_0, \vec{r})\, d\vec{r}$$

Rešitev

$$u(\vec{r}_0) = \oint_{\partial D} g(\vec{r}) \underbrace{\frac{\partial G}{\partial n}(\vec{r}_0, \vec{r})}_{\substack{\text{odvod } G \text{ glede na } \vec{r} \\ \perp \text{ na } \partial D}} d\vec{r}$$

Kako poiskati Greenovo fjo območja

$D^{\text{območje}} \subseteq \mathbb{R}^2$, iščemo $G(\vec{r}_0, \vec{r})$



(i) $G(\vec{r}, \vec{r}_0) = 0 \quad \forall \vec{r} \in \partial D$

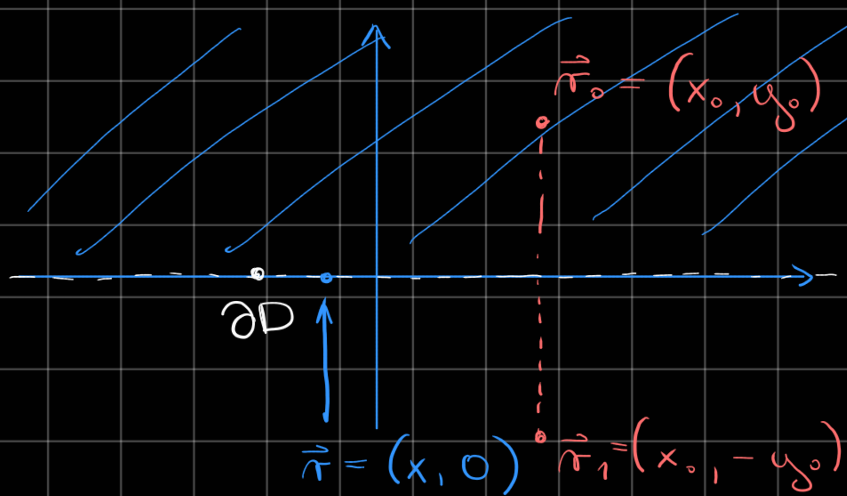
(ii) $G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_0| + \underbrace{\varphi(\vec{r}, \vec{r}_0)}_{\text{harmonska na } D}$

$\Updownarrow$

$\Delta_{\vec{r}} G = \delta(\vec{r} - \vec{r}_0)$

(2) Poišči Greenovo fjo za območje $D = \{(x,y) \in \mathbb{R}^2;\ y \geq 0\}$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_0| \underbrace{- \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_1|}_{\text{želimo dodati tako fjo, da bo na robu } \partial D = \emptyset}$$

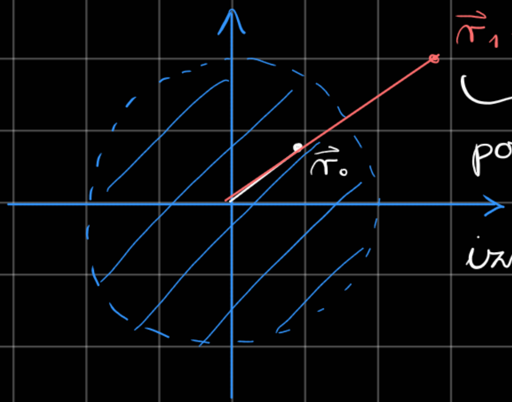


Ali je $G(\vec{r}, \vec{r}_0) = 0 \quad \forall \vec{r} \in \partial D$

$\vec{r}\ (x, 0)$

$$G((x,0),(x_0,y_0)) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + y_0^2} - \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (-y_0)^2} = 0$$

③ Poišči Greenovo fjo za $B(0,1)$



$\underbrace{\vec{r}_1 = \frac{\vec{r}_0}{|\vec{r}_0|^2}}$

podano, lahko pa izračunaš $\vec{r}_1 = \alpha \vec{r}_0$

Prezrcalimo čez rob, da bi bila $G(\vec{r}, \vec{r}_0)$ na robu enaka $\emptyset$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_0| - \underbrace{\frac{1}{2\pi} \ln\left(|\vec{r}_0||\vec{r} - \vec{r}_0|\right)}$$

harmonična na $B(0,1)$, ampak ne zunaj!

↳ pol je v $\vec{r}_1$, ki pa $\vec{r}_1 \notin B(0,1)$

Ali je $G(\vec{r}, \vec{r}_0) = 0 \quad \forall \vec{r} \in \underbrace{\partial B(0,1)}$?

$$\Updownarrow$$

$$|\vec{r}| = 1$$

Naj bo $|\vec{r}| = 1$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{2\pi} \ln \frac{|\vec{r} - \vec{r}_0|}{|\vec{r}_0||\vec{r} - \vec{r}_0|} \stackrel{?}{=} 0$$

Vrednost v logaritmu mora biti enako 1

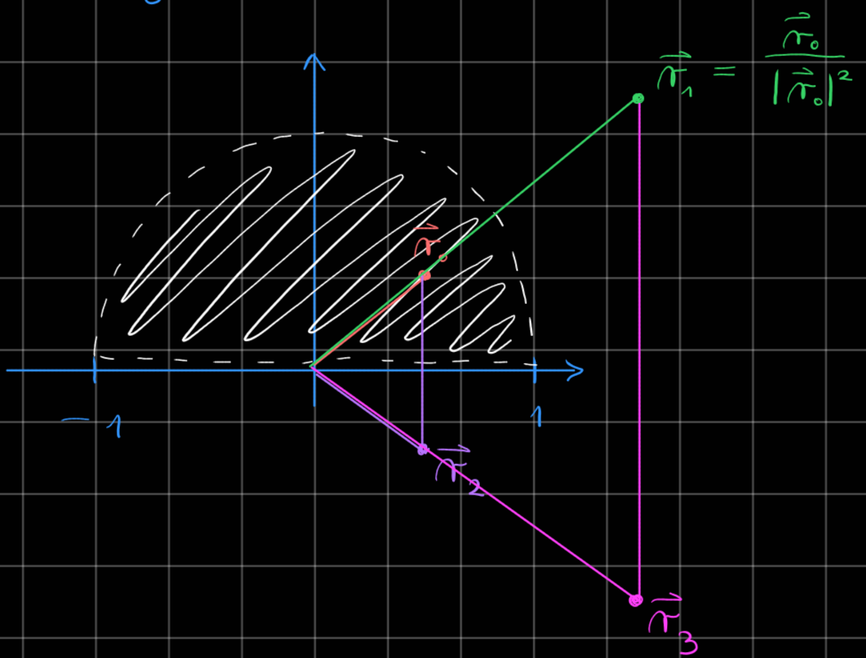
$$|\vec{r}-\vec{r}_0|^2 = |\vec{r}_0|^2 |\vec{r} - \vec{r}_0|^2$$

$$(\vec{r}-\vec{r}_0)(\vec{r}-\vec{r}_0) = |\vec{r}_0|^2\left(\vec{r} - \frac{\vec{r}_0}{|\vec{r}_0|^2}\right)\left(\vec{r} - \frac{\vec{r}_0}{|\vec{r}_0|^2}\right)$$

$$1 - 2\vec{r}\vec{r}_0 + |\vec{r}_0|^2 = |\vec{r}_0|^2\left(1 - \frac{2\vec{r}\vec{r}_0}{|\vec{r}_0|^2} + \frac{|\vec{r}_0|^2}{|\vec{r}_0|^4}\right)$$

$$= |\vec{r}_0|^2 - 2\vec{r}\vec{r}_0 + 1$$

3 Poišči Greenovo funkcijo na $D = \{(x,y) \in \mathbb{R}^2,\ y>0$ in $x^2+y^2>1\}$



$$G(\vec{r},\vec{r}_0) = \overset{①}{\frac{1}{2\pi}\ln|\vec{r}-\vec{r}_0|} - \overset{②}{\frac{1}{2\pi}\ln\left(|\vec{r}_0||\vec{r}-\vec{r}_1|\right)} - \overset{③}{\frac{1}{2\pi}\ln|\vec{r}-\vec{r}_2|}$$

$$\underbrace{+ \overset{④}{\frac{1}{2\pi}\ln\left(|\vec{r}_0||\vec{r}-\vec{r}_3|\right)}}$$

uvedemo, da se slika v 0 pri ukrivljenem delu polkroga

$$G(\vec{r}, \vec{r}_0) = 1 - 2 - 3 + 4$$

i) na robu polkroga

$$G(\vec{r}, \vec{r}_0) = \cancel{(1 - 2)} + \cancel{(-3 + 4)}$$

ii) $\vec{r}$ na osi

$$G(\vec{r}, \vec{r}_0) = 1 - 3 - 2 + 4$$